ARTICLE

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An open letter concerning

Alexander duality for monomial ideals and their resolutions

Dear Reader,

This article was submitted to Journal of Pure and Applied Algebra on December 15, 1998, and it was rejected with a very short report about eight months later, the cited reason being that it was too long for its content. By the time I received that overdue rejection, I was nearly done writing a sequel,

> Ezra Miller, *The Alexander duality functors and local duality with monomial support*, Journal of Algebra **231** (2000), 180–234.

which contained more general results. The sequel has been well-cited, but the current article was already on the arXiv (math.AC/9812095), and according to Google Scholar it has also been well-cited. In fact, this article has been cited more than most of my others—as much or more, for example, than my articles in Journal of the American Mathematical Society and Duke Mathematical Journal. It seemed a shame that what is apparently a useful article should languish in eternal semipublication, so I submitted it to Rejecta Mathematica.

Why is this article useful? It is more concrete than its sequel: more examples, more illustrations, and fewer functors. The article contains no known errors and no known uncited rederivations of earlier work; in fact, subsequent work (by other authors as well as in its sequel) has confirmed the results herein by independent methods many times over. The article is unchanged from the version submitted to Journal of Pure and Applied Algebra.

Ezra Miller
25 February 2008

Alexander Duality for Monomial Ideals and Their Resolutions

Ezra Miller

Abstract

Alexander duality has, in the past, made its way into commutative algebra through Stanley-Reisner rings of simplicial complexes. This has the disadvantage that one is limited to squarefree monomial ideals. The notion of Alexander duality is generalized here to arbitrary monomial ideals. It is shown how this duality is naturally expressed by Bass numbers, in their relations to the Betti numbers of a monomial ideal and its Alexander dual. The effect of Alexander duality on free resolutions is studied in the context of cellular resolutions. Relative cohomological constructions on cellular complexes are shown to relate cellular free resolutions of a monomial ideal to free resolutions of its Alexander dual ideal.

Introduction

Alexander duality in its most basic form is a relation between the homology of a simplicial complex Γ and the cohomology of another simplicial complex $\Gamma^\vee$, called the *dual* of Γ. Recently there has been much interest in the consequences of this relation when applied to the monomial ideals which are the Stanley-Reisner ideals I_Γ and $I_{\Gamma^\vee}$ for the given simplicial complex and its Alexander dual. This has the limitation that Stanley-Reisner ideals are always squarefree. The first aim of this paper is to define Alexander duality for arbitrary monomial ideals and then generalize some of the relations between I_Γ and $I_{\Gamma^\vee}$. A second goal is to demonstrate that Bass numbers are the proper vessels for the translation of Alexander duality into commutative algebra. The final goal is to reveal the connections between Alexander duality and the recent work on cellular resolutions.

There are two "minimal" ways of describing an arbitrary monomial ideal: via the minimal generators or via the (unique) irredundant irreducible decomposition. Given a monomial ideal I, Definition 1.5 describes a method for producing another monomial ideal $I^\vee$ whose minimal generators correspond to the irredundant irreducible components of I. Miraculously, this is enough to guarantee that the minimal generators of I correspond to the irreducible components of $I^\vee$. It is particularly easy to verify that this reversal of roles takes place for the squarefree ideals $I = I_\Gamma$ and $I^\vee = I_{\Gamma^\vee}$ above (Proposition 1.10). A connection with linkage and canonical modules is described in Theorem 2.1.

One can also deal with Alexander duality as a combinatorial phenomenon, thinking of Γ as an order ideal in the lattice of subsets of $\{1, \ldots, n\}$. The Alexander dual $\Gamma^\vee$ is then given by the complement of the order ideal, which gives an order ideal in the opposite lattice. For squarefree monomial ideals all is well since the only monomials we care about are represented precisely by the lattice of subsets of $\{1, \ldots, n\}$. For general monomial ideals we instead consider the larger lattice $\mathbb{Z}^n$, by which we mean the poset with its natural partial order $\preceq$. Then a monomial ideal I can be regarded as a dual order ideal in $\mathbb{Z}^n$, and $I^\vee$ is constructed (roughly) from the complementary set of lattice points, which is an order ideal—see Definition 2.9. It is Theorem 2.13 which proves the equivalence of the two definitions.

Bass numbers first assert themselves in Section 3. Their relations to Betti numbers for monomial modules (Corollary 3.6 and Theorem 3.12) are derived as consequences of graded local duality and Alexander duality (in its avatar as lattice duality in $\mathbb{Z}^n$). The Bass-Betti relations are then massaged to equate the localized Bass numbers of I (Definition 4.8) with the Betti numbers of $I^\vee$ in the first of the two central results of this paper, Theorem 4.10. Theorem 2.13 is then recovered as a special case of this main result, which also finds an application to inequalities between the Betti numbers of dual ideals (Theorem 4.13) generalizing those for squarefree ideals in [2].

The extension of Alexander duality to resolutions is accomplished in Sections 5 and 6. A new canonical and geometric resolution, the *cohull resolution* is constructed in Definition 5.15. It should be thought of as Alexander dual to the *hull resolution* of [4] (which is similarly canonical and geometric). Roughly speaking, the cohull resolution is constructed from the irreducible components instead of the minimal generators. The cohull resolution owes its existence to the second central result of the paper, Theorem 5.8, which is a more general result on duality for cellular resolutions. Its proof, which is resolutely algebraic, is the content of Section 6. The idea is to deform an ideal into its dual step by step via Definition 6.1 and keep track of the deformations on cellular resolutions (Theorem 6.9). The final step, taken in Theorem 6.11, is to check the effect of the deformations on the homology of the resolutions.

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1 Definitions and basic properties

For notation, let S be the $\mathbb{Z}^n$-graded k-algebra $k[x_1, \ldots, x_n] \subseteq T := S[x_1^{-1}, \ldots, x_n^{-1}]$, where k is a field and $n \geq 2$. If $A \subseteq T$ is any subset, $\langle a \mid a \in A\rangle$ will denote the S-submodule generated by the elements in A, and it may also be regarded as an ideal if $A \subseteq S$. The maximal $\mathbb{Z}^n$-graded ideal $\langle x_1, \ldots, x_n\rangle$ of S will be denoted by $\mathfrak{m}$. Each (Laurent) monomial in T is specified uniquely by a single vector $\mathbf{a} = (a_1, \ldots, a_n) = \sum_i a_i\mathbf{e}_i \in \mathbb{Z}^n$, while each irreducible monomial ideal is specified uniquely by a vector $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{N}^n$, so the notation

$$x^{\mathbf{a}} = x_1^{a_1} \cdots x_n^{a_n} \qquad \text{and} \qquad \mathfrak{m}^{\mathbf{b}} = \langle x_i^{b_i} \mid b_i \geq 1\rangle$$

will be used to highlight the similarity. The $\mathbb{Z}^n$-graded prime ideals, which are precisely the monomial prime ideals, are indexed by faces of the $(n-1)$-simplex $\Delta := 2^{\{1,\ldots,n\}}$ with vertices $1, \ldots, n$. Identifying a face $F \in \Delta$ with its characteristic vector in $\mathbb{Z}^n$, the monomial prime corresponding to F may be written with the above notation as $\mathfrak{m}^F$. Note, in particular, that $\mathfrak{m}^{\mathbf{b}}$ need not be an artinian ideal, just as $x^{\mathbf{a}}$ need not have full support. In fact, $\mathfrak{m}^{\mathbf{b}}$ is $\mathfrak{m}^{\sqrt{\mathbf{b}}}$-primary, where $\sqrt{\mathbf{b}} \in \Delta$ is the face representing the support of $\mathbf{b}$; that is, $\sqrt{\mathbf{b}}$ has i^{th} coordinate 1 if $b_i \geq 1$ and 0 otherwise. With this notation, taking radicals can be expressed as $\sqrt{\mathfrak{m}^{\mathbf{b}}} = \mathfrak{m}^{\sqrt{\mathbf{b}}}$.

All modules N and homomorphisms of such will be $\mathbb{Z}^n$-graded, so that $N = \bigoplus_{\mathbf{a}\in\mathbb{Z}^n} N_{\mathbf{a}}$. In addition, any module that is isomorphic to a submodule of T *as a $\mathbb{Z}^n$-graded module* will, if it is convenient, be freely identified with that submodule of T. For instance, the principal ideal generated by $x_1 \cdots x_n$ can be identified with the module $S[-\mathbf{1}]$, where $\mathbf{1} = (1, \ldots, 1) \in \mathbb{Z}^n$ and $N[\mathbf{a}]_{\mathbf{b}} = N_{\mathbf{a}+\mathbf{b}}$ for $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^n$. In this paper, ideals will all be proper monomial ideals, and the symbol I will always denote such an ideal. The vector $\mathbf{a}_I$ will denote the exponent on the least common multiple of the minimal generators of I.

Before making the definition of Alexander dual ideal, the next few results make sure that the exponents used to define the set $\mathrm{Irr}\,(I)$ of irredundant irreducible components of I are $\preceq \mathbf{a}_I$. For the next two results, let Λ denote the set of irreducible ideals containing I.

Lemma 1.1 *If $\mathfrak{m}^{\mathbf{b}} \in \mathrm{Irr}\,(I)$ then $\mathfrak{m}^{\mathbf{b}}$ is minimal (under inclusion) in Λ.*

Proof: Suppose $\mathfrak{m}^{\mathbf{b}} \neq \mathfrak{m}^{\mathbf{c}}$ and that $\mathfrak{m}^{\mathbf{b}} \supseteq \mathfrak{m}^{\mathbf{c}} \in \Lambda$. If now $I = \mathfrak{m}^{\mathbf{b}} \cap I'$ for some ideal I' then also $I = \mathfrak{m}^{\mathbf{c}} \cap I'$, whence $\mathfrak{m}^{\mathbf{b}} \notin \mathrm{Irr}\,(I)$. □

Proposition 1.2 *If $\mathfrak{m}^{\mathbf{b}} \in \mathrm{Irr}\,(I)$ then for each $i \in \sqrt{\mathbf{b}}$ there is a minimal generator $x^{\mathbf{c}}$ of I with $b_i = c_i$.*

Proof: Suppose $\mathfrak{m}^{\mathbf{b}} \in \mathrm{Irr}\,(I)$ but the conclusion does not hold. Then given any minimal generator $x^{\mathbf{c}}$ of I, either $b_{i'} \leq c_{i'}$ for some $i \neq i' \in \sqrt{\mathbf{b}}$, or else $b_i < c_i$. In either case, $x^{\mathbf{c}} \in \mathfrak{m}^{\mathbf{b}+\mathbf{e}_i}$, where $\mathbf{e}_i$ is the i^{th} unit vector in $\mathbb{Z}^n$. Then $\mathfrak{m}^{\mathbf{b}+\mathbf{e}_i} \supseteq I$, contradicting the minimality of $\mathfrak{m}^{\mathbf{b}}$ in Λ. □

Corollary 1.3 *For any $\mathfrak{m}^{\mathbf{b}} \in \mathrm{Irr}\,(I)$ we have $\mathbf{b} \preceq \mathbf{a}_I$.* □

The following notation will be very convenient in the definition and handling of Alexander duality. For any vector $\mathbf{a} \in \mathbb{Z}^n$ and any face $F \in \Delta$, let $\mathbf{a} \cdot F$ denote the restriction of $\mathbf{a}$ to F:

$$(\mathbf{a} \cdot F)_i \;=\; \begin{cases} a_i & \text{if } i \in F \\ 0 & \text{otherwise} \end{cases} .$$

This operation may also be thought of as the coordinatewise product of $\mathbf{a}$ and F. If, in addition, $\mathbf{0} \preceq \mathbf{b} \preceq \mathbf{a}$, define $\mathbf{b}^{\mathbf{a}}$ to be the vector whose i^{th} coordinate is $a_i + 1 - b_i$ if $b_i \geq 1$ and 0 otherwise; more compactly,

$$\mathbf{b}^{\mathbf{a}} \;=\; (\mathbf{a} + \mathbf{1} - \mathbf{b}) \cdot \sqrt{\mathbf{b}} \;=\; (\mathbf{a} + \mathbf{1}) \cdot \sqrt{\mathbf{b}} - \mathbf{b}\,,$$

where $\sqrt{\mathbf{b}}$ is the support of $\mathbf{b}$, as above. The next result is a first indication of the utility of $\mathbf{b}^{\mathbf{a}}$ when applied to irreducible ideals $\mathfrak{m}^{\mathbf{b}}$.

Proposition 1.4 *If $\mathbf{0} \preceq \mathbf{b}, \mathbf{c} \preceq \mathbf{a}$ then $\mathfrak{m}^{\mathbf{b}} \supseteq \mathfrak{m}^{\mathbf{c}}$ if and only if $\mathbf{b}^{\mathbf{a}} \succeq \mathbf{c}^{\mathbf{a}}$.*

Proof: The condition $\mathfrak{m}^{\mathbf{b}} \supseteq \mathfrak{m}^{\mathbf{c}}$ is equivalent to the combination of (i) $\sqrt{\mathbf{b}} \succeq \sqrt{\mathbf{c}}$ and (ii) $\mathbf{b} \cdot \sqrt{\mathbf{c}} \preceq \mathbf{c}$. Now consider the inequalities in the following chain:

$$\mathbf{b}^{\mathbf{a}} \quad = \quad (\mathbf{a}+\mathbf{1}-\mathbf{b}) \cdot \sqrt{\mathbf{b}} \quad \succeq \quad (\mathbf{a}+\mathbf{1}-\mathbf{b}) \cdot \sqrt{\mathbf{c}} \quad \succeq \quad (\mathbf{a}+\mathbf{1}-\mathbf{c}) \cdot \sqrt{\mathbf{c}} \quad = \quad \mathbf{c}^{\mathbf{a}}\,.$$

The left inequality is equivalent to (i) since $\mathbf{a}+\mathbf{1}-\mathbf{b}$ has full support, and the right inequality is equivalent to (ii) since $\mathbf{c} \cdot \sqrt{\mathbf{c}} = \mathbf{c}$. It remains only to show that $\mathbf{b}^{\mathbf{a}} \succeq \mathbf{c}^{\mathbf{a}}$ implies both inequalities, and this can be checked coordinatewise. If $c_i = 0$, then both inequalities become trivial; if $c_i > 0$ then $b_i > 0$, and the left inequality becomes an equality while the right inequality becomes $(\mathbf{b}^{\mathbf{a}})_i = a_i + 1 - b_i \geq a_i + 1 - c_i = (\mathbf{c}^{\mathbf{a}})_i\,$. □

Corollary 1.3 clears the way for the main definition of this paper:

Definition 1.5 (Alexander duality) *Given an ideal I and $\mathbf{a} \succeq \mathbf{a}_I$, the* Alexander dual ideal $I^{\mathbf{a}}$ with respect to $\mathbf{a}$ *is defined by*

$$I^{\mathbf{a}} \quad = \quad \langle x^{\mathbf{b}^{\mathbf{a}}} \mid \mathfrak{m}^{\mathbf{b}} \in \operatorname{Irr}(I) \rangle.$$

For the special case when $\mathbf{a} = \mathbf{a}_I$, let $I^\vee = I^{\mathbf{a}_I}$.

Remark 1.6 (i) We will never have occasion to take an Alexander dual of the ideal $\mathfrak{m}$, so $\mathfrak{m}^{\mathbf{a}}$ will retain its original definition.

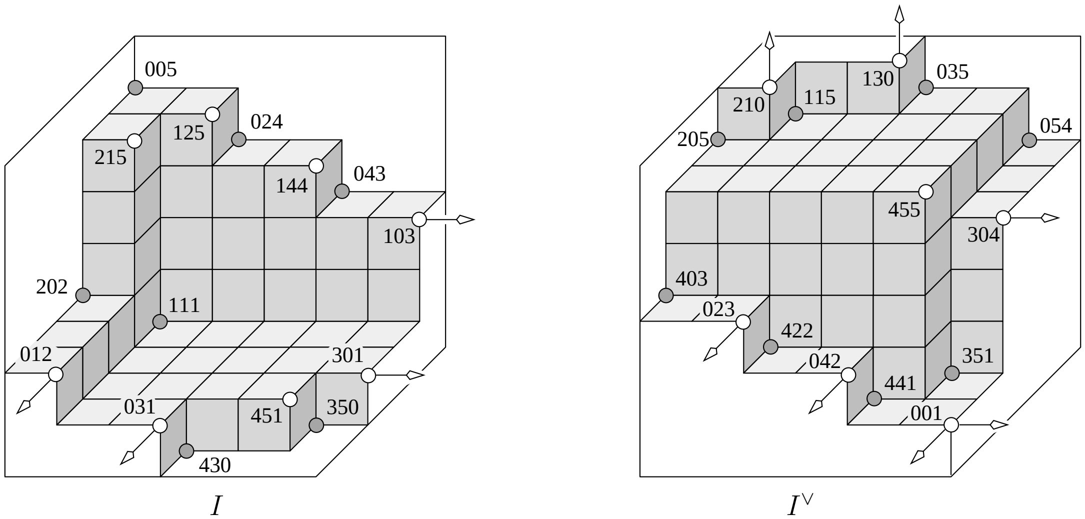
(ii) The dual $I^{\mathbf{a}}$ with respect to any $\mathbf{a} \succeq \mathbf{a}_I$ depends only on $\mathbf{a} \cdot \sqrt{\mathbf{a}_I}$. This is because $\mathbf{b}$ and $\mathbf{a} \cdot \sqrt{\mathbf{b}}$ determine $\mathbf{b}^{\mathbf{a}}$, and $\mathbf{a} \cdot \sqrt{\mathbf{b}} = (\mathbf{a} \cdot \sqrt{\mathbf{a}_I}) \cdot \sqrt{\mathbf{b}}$ for all of the relevant $\mathbf{b}$ by Corollary 1.3. In particular, $I^\vee = I^{\mathbf{1}}$ if I is squarefree.

(iii) $I^\vee$ is not gotten by taking the depolarization of the Alexander dual of the polarization of I (see [14], Chapter II for polarization). For instance, when $I = \langle x^2, xy, y^2 \rangle$, the polarization is $I_{polar} = \langle x_1x_2, x_1y_1, y_1y_2 \rangle$, whose canonical Alexander dual is $I_{polar}^\vee = \langle x_1y_1, x_1y_2, x_2y_1 \rangle$. Removing the subscripts on x and y then yields the principal ideal $\langle xy \rangle$, whereas $I^\vee = \langle xy^2, x^2y \rangle$.

Proposition 1.7 *The set of generators for $I^{\mathbf{a}}$ given by the definition is minimal. More generally, suppose $\mathbf{a} \succeq \mathbf{a}_I$ and Λ is a collection of integer vectors $\preceq \mathbf{a}$ such that $I = \bigcap_{\mathbf{b} \in \Lambda} \mathfrak{m}^{\mathbf{b}}$. Then $I^{\mathbf{a}} = \langle x^{\mathbf{b}^{\mathbf{a}}} \mid \mathbf{b} \in \Lambda \rangle$, and the intersection determined by Λ is irredundant if and only if the set of generators for $I^{\mathbf{a}}$ is minimal.*

Proof: This follows from Corollary 1.3 and Proposition 1.4. □

Example 1.8 Let $n = 3$, so that $S = k[x,y,z]$. Figure 1 lists the minimal generators and irredundant irreducible components of an ideal $I \subseteq S$ and its dual $I^\vee$ with respect to $\mathbf{a}_I$. The (truncated) "staircase diagrams" representing the monomials not in these ideals are also rendered in Figure 1. In fact, the staircase diagram for $I^\vee$ is gotten by literally turning the staircase diagram for I upside-down (the reader is encouraged to try this). Notice that the support of a minimal generator of I is equal to the support of the corresponding irreducible component of $I^\vee$. □



$$\begin{aligned} I &= \langle z^5, x^2z^2, x^4y^3, x^3y^5, y^4z^3, y^2z^4, xyz\rangle \\ &= \langle x^2,y,z^5\rangle \cap \langle y,z^2\rangle \cap \langle y^3,z\rangle \cap \langle x^4,y^5,z\rangle \cap \langle x^3,z\rangle \cap \langle x,z^3\rangle \cap \langle x,y^4,z^4\rangle \cap \langle x,y^2,z^5\rangle \end{aligned}$$

$$\mathbf{a} := \mathbf{a}_I = (4,5,5)$$

$$\begin{aligned} I^\vee &= \langle z\rangle \cap \langle x^3,z^4\rangle \cap \langle x,y^3\rangle \cap \langle x^2,y\rangle \cap \langle y^2,z^3\rangle \cap \langle y^4,z^2\rangle \cap \langle x^4,y^5,z^5\rangle \\ &= \langle x^3y^5z, y^5z^4, y^3z^5, xyz^5, x^2z^5, x^4z^3, x^4y^2z^2, x^4y^4z\rangle. \end{aligned}$$

Figure 1: The truncated staircase diagrams, minimal generators, and irredundant irreducible components for I and $I^\vee$. Black lattice points are generators, and white lattice points indicate irreducible components. The numbers are to be interpreted as vectors, e.g. 205 = (2,0,5). The arrows attached to a white lattice point indicate the directions in which the component continues to infinity; it should be noted that a white point has a zero in some coordinate precisely when it has an arrow pointing in the corresponding direction.

Example 1.9 Let Σ_n denote the symmetric group on $\{1,\ldots,n\}$ and $\mathbf{c} = (1,2,\ldots,n) \in \mathbb{N}^n$. The ideal $I = \langle x^{\sigma(\mathbf{c})} \mid \sigma \in \Sigma_n\rangle$ is the *permutahedron ideal* determined by $\mathbf{c}$, introduced in [4], Example 1.9. The results of Example 5.22 below imply that the canonical Alexander dual is the *forest ideal*, which is generated by $2^n - 1$ monomials: $I^\vee = \langle (x^F)^{n-|F|+1} \mid \emptyset \neq F \in \Delta\rangle$. For instance, when $n = 3$,

$$\begin{aligned} I &= \langle xy^2z^3, xy^3z^2, x^2yz^3, x^2y^3z, x^3yz^2, x^3y^2z\rangle \\ I^\vee &= \langle xyz, x^2y^2, x^2z^2, y^2z^2, x^3, y^3, z^3\rangle. \end{aligned}$$

The quotient of S by the forest ideal has the same dimension (over k) as the algebra $\mathcal{A}_n$ generated by the Chern 2-forms of the tautological line bundles over a flag manifold (see [10] and [13]). More precisely, the standard monomials of $I^\vee$, which are known to be in bijection with the forests on n labelled vertices, are shown in [10] to be a k-basis of $\mathcal{A}_n$. The minimal free resolution of $I^\vee$ is obtained in Example 5.22, below. □

Recall that for a simplicial complex $\Gamma \subseteq \Delta$ the *Stanley-Reisner ideal* I_Γ of Γ is defined by the nonfaces of Γ:

$$I_\Gamma \;\;=\;\; \langle x^F \mid F \not\in \Gamma \rangle,$$

and the *Alexander dual simplicial complex* $\Gamma^\vee$ consists of the complements of the nonfaces of Γ:

$$\Gamma^\vee \;\;=\;\; \{F \in \Delta \mid \overline{F} \not\in \Gamma\},$$

where $\overline{F} = \{1, \ldots, n\} \setminus F$. Recall also that I_Γ may be equivalently described as

$$I_\Gamma \;\;=\;\; \bigcap_{\overline{F} \in \Gamma} \mathfrak{m}^F,$$

since $\mathfrak{m}^F \supseteq I \;\Leftrightarrow\; F$ has at least one vertex in each nonface of $\Gamma \;\Leftrightarrow\; \overline{F}$ is missing at least one vertex from each nonface of $\Gamma \;\Leftrightarrow\; \overline{F}$ is a face of Γ. Applying Definition 1.5 to the latter characterization of I_Γ yields:

Proposition 1.10 *For a simplicial complex $\Gamma \subseteq \Delta$ we have $I_\Gamma^\vee = I_{\Gamma^\vee}$.*

Proof: Observe that $\mathbf{b^1} = \mathbf{b}$ if $\mathbf{b} \in \{0,1\}^n$, and use Proposition 1.7 along with Remark 1.6(ii). We get $I_\Gamma^\vee \;=\; \langle x^F \mid \overline{F} \in \Gamma \rangle \;=\; \langle x^F \mid F \not\in \Gamma^\vee \rangle \;=\; I_{\Gamma^\vee}$. □

Thus, as promised, Definition 1.5 generalizes to arbitrary monomial ideals the definition of Alexander duality for squarefree monomial ideals. The connection with the squarefree case is never lost, however, because the general definition does the same thing to the zero-set of I as the squarefree definition does:

Proposition 1.11 *Taking Alexander duals commutes with taking radicals: $\sqrt{I^\vee} = \sqrt{I}^{\,\vee}$.*

Proof: Since $\mathbf{0} \preceq \mathbf{b} \preceq \mathbf{a}_I$ whenever $\mathfrak{m}^{\mathbf{b}} \in \mathrm{Irr}(I)$, the equality $\sqrt{\mathbf{b}} = \sqrt{\mathbf{b}^{\mathbf{a}_I}}$ follows from the definitions. Thus,

$$\begin{aligned}\sqrt{I^\vee} \;&=\; \langle x^{\sqrt{\mathbf{b}}} \mid \mathfrak{m}^{\mathbf{b}} \in \mathrm{Irr}(I)\rangle \\ &=\; \langle x^F \mid \mathfrak{m}^F \text{ is minimal among primes containing } I\rangle \\ &=\; \sqrt{I}^{\,\vee},\end{aligned}$$

the last equality using again the facts mentioned in the first line of the proof of Proposition 1.10. □

The notion of Alexander duality sheds light on the interconnections between some of the developments in [3], [4], and [15] concerning cellular resolutions and (co)generic monomial ideals. To begin with, consider the following condition on a set of vectors $\{\mathbf{b}^j = (b_1^j, \ldots, b_n^j) \in \mathbb{N}^n\}_{j=1}^r$:

$$b_i^j \geq 1 \;\Rightarrow\; b_i^j \neq b_i^{j'} \text{ for all } j' \neq j.$$

A *generic* ideal, as defined in [3], is an ideal whose minimal generators have exponent vectors satisfying the above condition; similarly, a *cogeneric* ideal, as defined in [15], is an ideal whose irredundant irreducible components have exponent vectors satisfying the above condition. Using Definition 1.5 the following is immediate (for any $\mathbf{a} \succeq \mathbf{a}_I$).

Proposition 1.12 *$I^{\mathbf{a}}$ is generic if and only if I is cogeneric.* □

Example 1.13 The ideal I in Example 1.8 is generic, while $I^\vee$ is cogeneric. □

The connections between the minimal resolutions of such ideals and cellular resolutions will be explored in Section 5.

Recall that the *Castelnuovo-Mumford regularity* and *initial degree* of a $\mathbb{Z}$-graded S-module L defined respectively by

$$\operatorname{reg}(L) := \max\{j \in \mathbb{Z} \mid \operatorname{Tor}_i(L,k)_{i+j} \neq 0\} \qquad \text{and} \qquad \operatorname{indeg}(L) := \min\{j \in \mathbb{Z} \mid L_j \neq 0\},$$

where L_j is the j^{th} component of L. The question was raised in [8], Question 10 whether there is a duality for possibly nonradical monomial ideals with the "amazing properties"

- $\mathrm{reg(I) - indeg(I) = dim(S/I^\vee) - depth(S/I^\vee)}$
- I is componentwise linear if and only if $S/I^\vee$ is sequentially Cohen-Macaulay

obeyed by Alexander duals in the squarefree case. Here, I is considered in its $\mathbb{Z}$-grading. Having defined a duality operation in this paper, some comments are obviously warranted.

First of all, it is unrealistic to expect the first property to extend to the arbitrary (nonradical) case since the right-hand side of the equation is bounded while the left-hand side is not, in general. For instance, if $d \in \mathbb{N}$ then $\operatorname{reg}(\mathfrak{m}^{d\cdot\mathbf{1}}) - \operatorname{indeg}(\mathfrak{m}^{d\cdot\mathbf{1}}) = n(d-1) - d$ while $(\mathfrak{m}^{d\cdot\mathbf{1}})^\vee = \langle x_1 \cdots x_n \rangle$ is Cohen-Macaulay. Nevertheless, there may be some class of ideals which behaves nicely under some kind of duality, not necessarily as defined here. As to whether or not such a class of ideals exists for the Alexander duality as defined here, such an investigation has not yet been made.

Unfortunately, the second property also fails for I and $I^{\mathbf{a}}$, for somewhat trivial reasons: almost every ideal has an artinian Alexander dual. Specifically, if I is arbitrary and $x = x_1 \cdots x_n$, then $S/(xI)^{\mathbf{a}}$ is artinian (for any $\mathbf{a} \succeq \mathbf{a}_I$), and hence Cohen-Macaulay. But the minimal free resolution of xI is just the shift by $\mathbf{1}$ of the minimal resolution of I. Thus every minimal resolution, be it componentwise linear or not, appears as the resolution of an ideal whose dual is a Cohen-Macaulay ideal; i.e. $S/I^{\mathbf{a}}$ Cohen-Macauly $\not\Rightarrow$ I componentwise linear.

One might still hope that the implication "I has a linear resolution $\Rightarrow$ $S/I^{\mathbf{a}}$ is sequentially Cohen-Macaulay" would hold, but even this fails, as the example below shows. The fundamental problem with the nonsquarefree case is that the $\mathbb{Z}$-degree of an element is not determined by the support of its $\mathbb{Z}^n$-graded degree, as it is with squarefree monomials. Thus an ideal might have a linear resolution while its generators have support sets of varying sizes, wreaking havoc with the equidimensionality required for the Cohen-Macaulayness of the dual. Even so, it would be very interesting to know what is the property Alexander dual to "sequentially Cohen-Macaulay"; perhaps this property could relax the requirements of componentwise linearity in a nice way.

Example 1.14 Let $I' = \langle ab, bc, cd\rangle \subseteq S = k[a,b,c,d]$ be the ideal of the "stick twisted cubic" simplicial complex spanned by the edges $\{b,d\}, \{b,c\}$, and $\{a,c\}$. It is readily checked that I' has a linear resolution: indeed, $(I')^\vee$ is the ideal of another stick twisted cubic, which is Cohen-Macaulay because the stick twisted cubic is connected and has dimension 1, so [6], Theorem 3 applies. Let

$$I = \mathfrak{m}I' = \langle a^2b, abc, acd, ab^2, b^2c, bcd, abc, bc^2, c^2d, abd, bcd, cd^2\rangle$$
$$I^\vee = \langle b^2d^2, b^2c^2, a^2c^2, abc^2d^2, a^2bcd^2, a^2b^2cd\rangle$$

with $\mathbf{a}_I = (2,2,2,2)$. Then I has a linear resolution by [8], Lemma 1, and we show that $S/I^\vee$ is not sequentially Cohen-Macaulay.

Recall that for a module N to be sequentially Cohen-Macaulay, we require that there exist a filtration $0 = N_0 \subset N_1 \subset \cdots \subset N_r = N$ such that N_i/N_{i-1} is Cohen-Macaulay for all $i \leq r$ and $\dim(N_{i+1}/N_i) > \dim(N_i/N_{i-1})$ for all $i < r$. It follows from the equidimensionality of N/N_{r-1} and the strict reduction of dimension in successive quotients that N_{r-1} is the top dimensional piece of N; i.e. N_{r-1} is the intersection of all primary components (of 0 in N) which have dimension $\dim(N)$. Thus it suffices to check that $S/I^\vee_{\text{top}}$ is not Cohen-Macaulay, where $I^\vee_{\text{top}} = \langle b^2d^2, b^2cd, abcd, b^2c^2, abc^2, a^2c^2\rangle$ is the intersection of all primary components of $I^\vee$ which have dimension $2 = \dim(S/I^\vee)$. $\square$

2 Alternate characterizations of the Alexander dual ideal

Definition 1.5 is quite satisfactory for the consequences just derived from it, but it can sometimes be inconvenient to work with. For instance, it is not obvious from the definition that $(I^{\mathbf{a}})^{\mathbf{a}} = I$, which is fundamental—see Corollary 2.14. For this and other applications, we set out now to find other characterizations of the Alexander dual ideal in Theorem 2.1 and in Definition 2.9 with Theorem 2.13. Along the way, an algebraic analogue of combinatorial lattice duality in $\mathbb{Z}^n$ is defined in Defintion 2.3.

First, a result relating Alexander duality to linkage (see [17], Appendix A.9 for a brief introduction to linkage, and references):

Theorem 2.1 *If* $\mathbf{a} \succeq \mathbf{a}_I$ *then* $(\mathfrak{m}^{\mathbf{a}+\mathbf{1}} : I^{\mathbf{a}}) = I + \mathfrak{m}^{\mathbf{a}+\mathbf{1}}$.

Proof: Let $\text{Min}(I^{\mathbf{a}})$ denote the exponents on the minimal generators of $I^{\mathbf{a}}$. Then $(\mathfrak{m}^{\mathbf{a}+\mathbf{1}} : I^{\mathbf{a}}) = \bigcap_{\mathbf{b}\in \text{Min}(I^{\mathbf{a}})}(\mathfrak{m}^{\mathbf{a}+\mathbf{1}} : x^{\mathbf{b}})$. But $x^{\mathbf{c}} \cdot x^{\mathbf{b}} \in \mathfrak{m}^{\mathbf{a}+\mathbf{1}} \Leftrightarrow \mathbf{b}+\mathbf{c} \not\preceq \mathbf{a} \Leftrightarrow \mathbf{c} \not\preceq \mathbf{a}-\mathbf{b} \Leftrightarrow x^{\mathbf{c}} \in \mathfrak{m}^{\mathbf{a}+\mathbf{1}-\mathbf{b}}$. Thus, taking all intersections over $\mathbf{b} \in \mathrm{Min}\,(I^{\mathbf{a}})$,

$$\bigcap(\mathfrak{m}^{\mathbf{a+1}} : x^{\mathbf{b}}) \;=\; \bigcap \mathfrak{m}^{\mathbf{a+1-b}} \;=\; \bigcap\left(\mathfrak{m}^{\mathbf{b^a}} + \mathfrak{m}^{\mathbf{a+1}}\right) \;=\; \left(\bigcap \mathfrak{m}^{\mathbf{b^a}}\right) + \mathfrak{m}^{\mathbf{a+1}} \;=\; I + \mathfrak{m}^{\mathbf{a+1}}$$

since $(\mathbf{b^a})^{\mathbf{a}} = \mathbf{b}$ for all $\mathbf{b} \preceq \mathbf{a}$. □

Remark 2.2 Using Corollary 2.14, below, this theorem provides a useful way to compute the Alexander dual ideal, given a set of generators. Indeed, the generators for $I^{\mathbf{a}}$ are simply those generators of $(\mathfrak{m}^{\mathbf{a+1}} : I)$ whose exponents are $\preceq \mathbf{a}$. Using Definition 1.5 (and Corollary 2.14 again), this can also be construed as a method for computing irreducible components of I given a generating set for I, or vice versa.

Denoting the $\mathbb{Z}^n$-graded Hom functor by $\underline{\mathrm{Hom}}$, the next duality that comes into play is the k*-dual* $N^\wedge := \underline{\mathrm{Hom}}_k(N, k)$, which is a $\mathbb{Z}^n$-graded S-module with the grading $(N^\wedge)_{\mathbf{c}} = \mathrm{Hom}_k(N_{-\mathbf{c}}, k)$. It is a simple but very important observation that $T^\wedge \cong T$ as $\mathbb{Z}^n$-graded modules. This can be exploited: let $M \subseteq T$ be a submodule (the $\mathbb{Z}^n$-graded submodules of T are precisely the *monomial modules* of [4]). Taking the k-dual of the surjection $T \to T/M$ yields an injection $(T/M)^\wedge \to T^\wedge \cong T$. This makes $(T/M)^\wedge$ into a submodule of T which we call the T*-dual of* M and denote by M^T. If one thinks of the module M as a set of lattice points in $\mathbb{Z}^n$, then M^T can be thought of as the negatives of the lattice points in the complement of M; i.e. we can make the equivalent

Definition 2.3 *The* T-dual M^T *of a monomial module* $M \subseteq T$ *is defined by* $x^{-\mathbf{b}} \in M^T \Leftrightarrow x^{\mathbf{b}} \notin M$.

The equivalence with the earlier formulation can be seen simply by examining which $\mathbb{Z}^n$-graded pieces of M and M^T have dimension 1 over k and which have dimension 0. Observe the striking similarity of Definition 2.3 with definition of the dual simplicial complex: $\overline{F} \in \Gamma^\vee \Leftrightarrow F \notin \Gamma$. Here are some properties of the T-dual which will be used later (possibly without explicit reference). Note the similarity of (i)–(iii) to the laws governing complements, unions, and intersections.

Proposition 2.4 *Let* M *and* N *be submodules of* T*. Then*

(i)	$(M^T)^T = M$	*(v)*	$T/M^T = M^\wedge$
(ii)	$M \subseteq N \;\Leftrightarrow\; N^T \subseteq M^T$	*(vi)*	$(N/M)^\wedge = M^T/N^T$ if $M \subseteq N$
(iii)	$(M+N)^T = M^T \cap N^T$	*(vii)*	$(N/\,N\cap M)^\wedge = M^T/\,M^T\cap N^T$
(iv)	$M[\mathbf{a}]^T = M^T[-\mathbf{a}]$		

Proof: Statements (i)–(iv) follow from Definition 2.3, and (v) follows either from the definition and (i) or as a special case of (vi). To prove (vi) observe that $N/M = \ker(T/M \to T/N)$ so that $(N/M)^\wedge = \mathrm{coker}((T/N)^\wedge \to (T/M)^\wedge)$ and use the definition of T-dual. Finally, (vii) is just (vi) and (iii) applied to $(N+M)/M = N/\,N\cap M$. □

Definition 2.5 *Given a monomial ideal $I \subseteq S$ define the* Čech hull of I *in* T*:*

$$\widetilde{I} \ := \ \langle x^{\mathbf{b}} \mid \mathbf{b} \in \mathbb{Z}^n \text{ and } x^{\mathbf{b}^+} \in I \rangle \,,$$

where $\mathbf{b}^+ \in \mathbb{N}^n$ is, as usual, the join (componentwise maximum) of $\mathbf{b}$ and $\mathbf{0}$ in the order lattice $\mathbb{Z}^n$.

Proposition 2.6 *Taking the Čech hull commutes with finite intersections and sums. Furthermore,*

(i) $\widetilde{I}$ *is the largest monomial submodule of T whose intersection with S is equal to I.*
(ii) $\widetilde{I}$ *can be generated by (possibly infinitely many) monomials in T of degree $\preceq \mathbf{a}_I$.*
(iii) $\widetilde{I}^{\,T}$ *is generated in degrees $\preceq \mathbf{0}$.*

Proof: The first statement follows from (i) and the definitions.

(i) It is clear from the definition that $\widetilde{I}$ contains I; and if $x^{\mathbf{b}} \in \widetilde{I} \cap S$ then $\mathbf{b}^+ = \mathbf{b}$ whence $x^{\mathbf{b}} \in I$. Thus $\widetilde{I} \cap S = I$. On the other hand, if M is a monomial submodule of T satisfying $M \cap S = I$ and $x^{\mathbf{b}} \in M$, then $x^{\mathbf{b}^-} \cdot x^{\mathbf{b}} = x^{\mathbf{b}^+} \in M \cap S = I$, where $\mathbf{b}^- := \mathbf{b}^+ - \mathbf{b}$. Thus $M \subseteq \widetilde{I}$.

(ii) If $x^{\mathbf{b}} \in \widetilde{I}$ then $\mathbf{c} \preceq \mathbf{b}^+$ for some minimal generator $x^{\mathbf{c}}$ of I, whence $x^{\mathbf{c}-\mathbf{b}^-}$ is in $\widetilde{I}$, divides $x^{\mathbf{b}}$, and has exponent $\preceq \mathbf{a}_I$.

(iii) The following statement is precisely the T-dual to statement (i): $\widetilde{I}^{\,T}$ is the smallest submodule whose sum with $\widetilde{\mathfrak{m}}$ is equal to $I^{\,T}$. As $\widetilde{\mathfrak{m}}$ already contains all degrees $\not\preceq \mathbf{0}$, minimality of $\widetilde{I}^{\,T}$ implies that it is generated in degrees $\preceq \mathbf{0}$. □

Example 2.7 (i) Recall that for $F \in \Delta$, the complement $\{1, \ldots, n\} \setminus F$ is denoted by $\overline{F}$. Using this, the localization $S[x^{-\overline{F}}]$ is achieved by inverting the variables x_i for $i \notin F$. Now let $\mathbf{b} \succ \mathbf{0}$ and $F = \sqrt{\mathbf{b}}$. Then

$$\left(\widetilde{\mathfrak{m}^{\mathbf{b}}}\right)^T \ = \ \left(S[x^{-\overline{F}}]\right)[\mathbf{b} - F]\,.$$

To see this, first observe that if $\mathbf{c} \in \mathbb{N}^n$ then $x^{\mathbf{c}} \notin \mathfrak{m}^{\mathbf{b}} \ \Leftrightarrow \ \mathbf{c} \cdot F \preceq \mathbf{b} - F$. Therefore, if $\mathbf{c} \in \mathbb{Z}^n$ then $x^{\mathbf{c}} \notin \widetilde{\mathfrak{m}^{\mathbf{b}}} \ \Leftrightarrow \ \mathbf{c}^+ \preceq \mathbf{b} - F \ \Leftrightarrow \ \mathbf{c} \cdot F \preceq \mathbf{b} - F$. This last condition is equivalent to $-\mathbf{c} \cdot F \succeq F - \mathbf{b}$, and this occurs if and only if $x^{-\mathbf{c}} \in \left(S[x^{-\overline{F}}]\right)[\mathbf{b} - F]$.

(ii) For a special case, it follows that when $\mathbf{b} \succeq \mathbf{0}$, $\widetilde{\mathfrak{m}^{\mathbf{b}+\mathbf{1}}} \ = \ S[\mathbf{b}]^T$. □

Remark 2.8 Example 2.7(ii) is the reason for the name *Čech hull*: when $\mathbf{b} = \mathbf{0}$, we find that $\widetilde{\mathfrak{m}}$ is the kernel of the last map in the Čech complex on $x_1, \ldots, x_n$.

Definition 2.9 *For any monomial ideal I and $\mathbf{a} \succeq \mathbf{a}_I$, define*

$$I^{[\mathbf{a}]} \ := \ \widetilde{I}^{\,T}[-\mathbf{a}] \cap S.$$

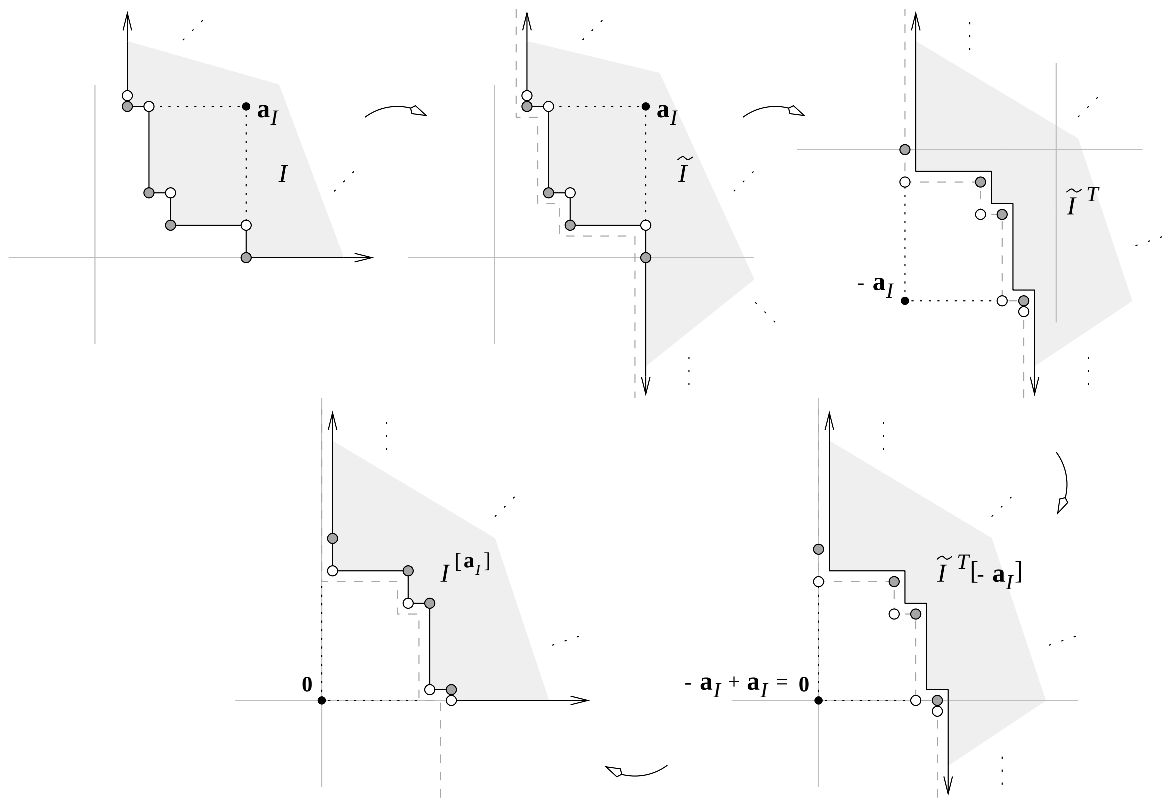


Figure 2

Example 2.10 Figure 2 is a schematic diagram depicting the transformation in stages from I to $I^{[\mathbf{a}_I]}$. The black and white dots shift by $\mathbf{1}$ from the penultimate stage to the last; they are left in place (with respect to the dark black dot and the dark dotted lines) for the rest of the transformation. This shift is the reason for the $\mathbf{1}$ in the definition of $\mathbf{b}^{\mathbf{a}}$, and it occurs because the flip-flop from $\widetilde{I}$ to $\widetilde{I}^{\,T}$ leaves a space of $\mathbf{1}$. The crux of this whole theory is that the "boundaries" of $\widetilde{I}$ and $\widetilde{I}^{\,T}$ have the same shape, but reversed, thus switching the roles of the black and white dots. This schematic may be helpful in parsing the proof of Theorem 2.13, below. $\square$

Lemma 2.11 $(I^{[\mathbf{a}]})^{\widetilde{}} = \widetilde{I}^{\,T}[-\mathbf{a}]$.

Proof: Let $M = \widetilde{I}^{\,T}[-\mathbf{a}]$. By Proposition 2.6(i), $(M \cap S)^{\widetilde{}} \supseteq M$ since their intersections with S are equal by definition. Thus $((M \cap S)^{\widetilde{}}\,)^T \subseteq M^T$, with equality in degrees $\preceq \mathbf{0}$. But $M^T = \widetilde{I}[\mathbf{a}]$ is generated in negative degrees by Proposition 2.6(ii), so that in fact $((M \cap S)^{\widetilde{}}\,)^T = M^T$. Taking T-duals of this equality gives the desired result. $\square$

The upshot is that I may be reconstructed from $I^{[\mathbf{a}]}$ via the same construction which produces $I^{[\mathbf{a}]}$ from I:

Proposition 2.12 $\mathbf{a}_{I^{[\mathbf{a}]}} \preceq \mathbf{a}$ *and* $I = (I^{[\mathbf{a}]})^{[\mathbf{a}]}$.

Proof: By Proposition 2.6(iii) $\widetilde{I}^T[-\mathbf{a}]$ is generated in degrees $\preceq \mathbf{a}$, so Lemma 2.11 implies that the same holds for $(I^{[\mathbf{a}]})\widetilde{\ }$. It is trivial to check that if any monomial module $M \subseteq T$ is generated in degrees $\preceq \mathbf{a}$ then so is $M \cap S$, because $\mathbf{a} \succeq \mathbf{0}$. Thus $\mathbf{a}_{I^{[\mathbf{a}]}} \preceq \mathbf{a}$, and, in particular, $(I^{[\mathbf{a}]})^{[\mathbf{a}]}$ is well-defined. Now

$$\begin{aligned} (I^{[\mathbf{a}]})^{[\mathbf{a}]} &= ((I^{[\mathbf{a}]})\widetilde{\ })^T[-\mathbf{a}] \cap S && \text{by definition} \\ &= (\widetilde{I}^T[-\mathbf{a}])^T[-\mathbf{a}] \cap S && \text{by the previous lemma} \\ &= \widetilde{I} \cap S && \text{by Proposition 2.4(iv) and (i)} \\ &= I. \end{aligned}$$

□

The real cause for introducing $I^{[\mathbf{a}]}$ is the next result, which may not be so unexpected at this point. It would seem that Theorem 2.13 makes the notation $I^{[\mathbf{a}]}$ superfluous, and it does; nevertheless, the notation will be retained for emphasis, to indicate that Sections 3 and 4 (and, in particular, Theorem 4.10) are logically independent from Theorem 2.13.

Theorem 2.13 $I^{\mathbf{a}} = I^{[\mathbf{a}]}$.

Proof: To simplify notation, declare that $\mathbf{b} \in \mathrm{Irr}(I)$ if $\mathfrak{m}^{\mathbf{b}} \in \mathrm{Irr}(I)$. For each $\mathbf{b} \in \mathrm{Irr}(I)$, let $S^{\mathbf{b}} = S[x^{-\overline{\sqrt{\mathbf{b}}}}]$ be the localization of S at the prime $\mathfrak{m}^{\sqrt{\mathbf{b}}}$. Then for each $\mathbf{b} \in \mathrm{Irr}(I)$ and any $\mathbf{c} \in \mathbb{N}^n$ we have the following two facts:

(i) $S^{\mathbf{b}}[-\mathbf{c}] \cong S^{\mathbf{b}}[-\mathbf{c} \cdot \sqrt{\mathbf{b}}\,]$ since multiplication by $x^{\mathbf{c}\cdot\overline{\sqrt{\mathbf{b}}}}$ is a $\mathbb{Z}^n$-graded automorphism of $S^{\mathbf{b}}[-\mathbf{c}]$.

(ii) $S \cap S^{\mathbf{b}}[-\mathbf{c} \cdot \sqrt{\mathbf{b}}\,] = S[-\mathbf{c} \cdot \sqrt{\mathbf{b}}\,]$. Indeed, this is equivalent to $\Big(\langle x^{\mathbf{c}\cdot\sqrt{\mathbf{b}}}\rangle \cdot S^{\mathbf{b}}\Big) \cap S = \langle x^{\mathbf{c}\cdot\sqrt{\mathbf{b}}}\rangle$, which holds because $\langle x^{\mathbf{c}\cdot\sqrt{\mathbf{b}}}\rangle \subseteq S$ is saturated with respect to $\langle x^{\overline{\sqrt{\mathbf{b}}}}\rangle$; i.e. $\Big(\langle x^{\mathbf{c}\cdot\sqrt{\mathbf{b}}}\rangle : x^{\overline{\sqrt{\mathbf{b}}}}\Big) = \langle x^{\mathbf{c}\cdot\sqrt{\mathbf{b}}}\rangle$.

Creating $I^{[\mathbf{a}]}$ from I in stages yields

$$\begin{aligned} & \widetilde{I} &&= \bigcap \widetilde{\mathfrak{m}^{\mathbf{b}}} && \text{by Proposition 2.6} \\ \Rightarrow\ & \widetilde{I}^T &&= \sum \left(\widetilde{\mathfrak{m}^{\mathbf{b}}}\right)^T && \text{by Proposition 2.4(iii)} \\ & &&= \sum S^{\mathbf{b}}[\mathbf{b} - \sqrt{\mathbf{b}}] && \text{by Example 2.7(i)} \\ \Rightarrow\ & \widetilde{I}^T[-\mathbf{a}] &&= \sum S^{\mathbf{b}}[-\mathbf{b}^{\mathbf{a}}] && \text{by (i) above, with } \mathbf{c} = \mathbf{a} + \sqrt{\mathbf{b}} - \mathbf{b} \\ \Rightarrow\ & S \cap \widetilde{I}^T[-\mathbf{a}] &&= \sum S[-\mathbf{b}^{\mathbf{a}}] && \text{by (ii) above, with } \mathbf{c} = \mathbf{b}^{\mathbf{a}} \end{aligned}$$

where the intersection and all of the summations are taken over all $\mathbf{b} \in \mathrm{Irr}(I)$. The last summation above is equal to $I^{\mathbf{a}}$ since each summand $S[-\mathbf{b}^{\mathbf{a}}]$ is just a principal ideal $\langle x^{\mathbf{b}^{\mathbf{a}}}\rangle$. □

Corollary 2.14 $(I^{\mathbf{a}})^{\mathbf{a}} = I$. *Furthermore,* $(\mathbf{b}^{\mathbf{a}})^{\mathbf{a}} = \mathbf{b}$, *so that*

$$I^{\mathbf{a}} = \bigcap \{\mathfrak{m}^{\mathbf{b}^{\mathbf{a}}} \mid x^{\mathbf{b}} \textit{ is a minimal generator of } I\}.$$

□

Remark 2.15 In general, one has $(I^\vee)^\vee \neq I$. However, in the special case when $I = \sqrt{I}$, it will always happen that $(I^\vee)^\vee = I$. This follows from an application of Corollary 2.14 to Remark 1.6(ii). The difference $\mathbf{a}_I - \mathbf{a}_{I^\vee}$ measures the extent to which $(I^\vee)^\vee \neq I$ fails, in the sense that $(I^\vee)^\vee = I[\mathbf{a}_I - \mathbf{a}_{I^\vee}] \cap S$. However $((I^\vee)^\vee)^\vee = I^\vee$, so that an ideal which is already an Alexander dual is maximal in some sense. It is unclear what the invariant $\mathbf{a}_I - \mathbf{a}_{I^\vee}$ means, in general, although the passage from I to $(I^\vee)^\vee$ can sometimes be thought of as a "tightening" that may resolve some amount of nonminimality in the hull resolution of [4]—see Example 5.27. See also Remark 5.9(ii) below for another occurrence of the invariant $\mathbf{a}_I - \mathbf{a}_{I^\vee}$.

The reader interested in cellular resolutions may wish to skip directly to Section 5, whose only logical dependence on Sections 3 and 4 is Proposition 3.11.

3 Bass numbers versus Betti numbers

Algebraically, Alexander duality is best expressed in terms of relations between Betti and Bass numbers (Definition 3.1), as evidenced by this section and the next. The principle behind this is that the T-duality of Section 2, which can be thought of as lattice duality in $\mathbb{Z}^n$, can also be interpreted (Corollary 3.6) as a manifestation of the same process that interchanges flat and injective modules (in the appropriate category). In Theorem 3.12 this results in equalities between Bass and Betti numbers of I. Though perhaps not so interesting a statement in its own right, Proposition 3.11 is the workhorse for the remainder of the paper—it is the *reason* everything else is true. It encapsulates simultaneously the relations between all of the dualities that enter into this paper: k- and T-duality, Alexander duality, linkage, local duality, and Matlis duality.

Definition 3.1 *The derived functors of the $\mathbb{Z}^n$-graded functor $\underline{\mathrm{Hom}}$ will be called $\underline{\mathrm{Ext}}$, and the left derived functor of $\otimes$, which is also $\mathbb{Z}^n$-graded, will be called $\underline{\mathrm{Tor}}$. For a module N define*

$$\begin{aligned} \mu_{i,\mathbf{b}}(N) &= \dim_k\Big(\underline{\mathrm{Ext}}\,^i_S(k,N)_\mathbf{b}\Big) \\ \beta_{i,\mathbf{b}}(N) &= \dim_k\Big(\underline{\mathrm{Tor}}\,^S_i(k,N)_\mathbf{b}\Big), \end{aligned}$$

the i^{th} Bass and Betti numbers of N in degree $\mathbf{b}$.

Remark 3.2 (i) In order to compute these derived functors in the category $\mathcal{M}$ of $\mathbb{Z}^n$-graded S-modules (see Proposition 3.3), we need to know that $\mathcal{M}$ has enough injective and projective modules, just as in the nongraded case. Of course, there are always free modules, so this takes care of the projectives; for injectives one can easily modify the proof of [5], Theorem 3.6.2 to fit the $\mathbb{Z}^n$-graded case.

(ii) If M is finitely generated then $\underline{\mathrm{Ext}}\,^\cdot(M,-) = \mathrm{Ext}\,^\cdot(M,-)$. In particular, summing the Betti or Bass numbers over all $\mathbf{b}$ (or all $\mathbf{b}$ with fixed $\mathbb{Z}$-degree) gives the same result as computing directly in the nongraded (or $\mathbb{Z}$-graded) case.

In what follows, we will need the notion of a *flat resolution in $\mathcal{M}$*. This is defined exactly like a free resolution, except that the resolving modules are required to be flat instead of free, where *flat* means acyclic for $\underline{\mathrm{Tor}}$ [18], Section 2.4. Recall that free and flat are equivalent for finitely generated S-modules; this is a simple consequence of the grading and Nakayama's lemma. However, non-finitely generated flat modules, such as localizations of S, may fail to be free, or even projective.

Proposition 3.3 *(i) $\underline{\mathrm{Ext}}^{\cdot}(M,N)$ can be calculated as the homology of the complexes obtained either by applying $\underline{\mathrm{Hom}}(-,N)$ to a projective resolution of M in $\mathcal{M}$ or by applying $\underline{\mathrm{Hom}}(M,-)$ to an injective resolution of N in $\mathcal{M}$.*
(ii) $\underline{\mathrm{Tor}}_{\cdot}(M,N)$ can be calculated as the homology of the complexes obtained by either tensoring with N a flat resolution of M in $\mathcal{M}$ or by tensoring with M a flat resolution of N in $\mathcal{M}$.

Proof: (i) Remark 3.2(i) above provides enough injectives to use [18], Definition 2.5.1, Example 2.5.3, and Exercise 2.7.4.

(ii) [18], Theorem 2.7.2 and Exercise 2.4.3. □

Lemma 3.4 $N^\wedge = \underline{\mathrm{Hom}}_S(N, S^\wedge)$.

Proof: [5], Proposition 3.6.16(c), whose proof holds just as easily in the $\mathbb{Z}^n$-graded case. □

The next theorem is the starting point for the comparison of Betti and Bass numbers. Its corollary, which carries out the lattice complementation, is fundamental to the rest of the results in this section.

Proposition 3.5 *For any module N, $\mu_{i,\mathbf{b}}(N) \;=\; \beta_{i,-\mathbf{b}}(N^\wedge)$.*

Proof: A module J is injective if and only if $J^\wedge$ is flat, because

$$\underline{\mathrm{Hom}}(-\,,\,J) \quad=\quad \underline{\mathrm{Hom}}(-\,,\,\underline{\mathrm{Hom}}(J^\wedge, S^\wedge)\,) \quad=\quad \underline{\mathrm{Hom}}(-\otimes J^\wedge\,,\,S^\wedge). \tag{1}$$

Indeed, the first term being an exact functor means that J is injective, while the last term being an exact functor means that $J^\wedge$ is flat, since $\underline{\mathrm{Hom}}(-,S^\wedge)$ is *a priori* a faithful exact functor. It follows that a complex $J^{\cdot} : 0 \to J^0 \to J^1 \to \cdots$ is an injective resolution of N in $\mathcal{M}$ if and only if $(J^{\cdot})^\wedge$ is a flat resolution of $N^\wedge$. Substituting k for $(-)$ in Equation (1) and applying Proposition 3.3 we get

$$\underline{\mathrm{Ext}}^i(k,N) \;\cong\; \underline{\mathrm{Tor}}_i(k,N^\wedge)^\wedge \tag{2}$$

from which the result follows at once. □

Corollary 3.6 $\mu_{i,\mathbf{b}}(T/M) = \beta_{i,-\mathbf{b}}(M^T)$ *for any monomial module $M \subseteq T$.* □

The next few results are preliminary to the theorems relating the Betti numbers of I to the Bass numbers of I (Theorem 3.12) and the Bass numbers of $I^{[\mathbf{a}]}$ (Theorem 4.10).

Proposition 3.7 *Let I be an ideal. Then*

$$\begin{array}{ll} (i) & \beta_{i,\mathbf{b}}(\widetilde{I}) = 0 \text{ if } \mathbf{b} \not\preceq \mathbf{a}_I. \\ (ii) & \beta_{i,\mathbf{b}}(\widetilde{I}) = 0 \text{ if } \mathbf{b} \not\succeq \mathbf{1}. \\ (iii) & \beta_{i,\mathbf{b}}(\widetilde{I}) = \beta_{i,\mathbf{b}}(I) \text{ if } \mathbf{1} \preceq \mathbf{b}. \end{array}$$

Proof: Given any submodule $M \subseteq T$, define for each $\mathbf{b} \in \mathbb{Z}^n$ the following simplicial subcomplex of Δ:

$$K_{\mathbf{b}}(M) \ = \ \{F \in \Delta \mid x^{\mathbf{b}-F} \in M\}.$$

It is a result of [9] and [12] (and extended to the case $M \subseteq T$ by [4]) that

$$\beta_{i,\mathbf{b}}(M) = \dim_k \widetilde{H}_i(K_{\mathbf{b}}(M); k),$$

the dimension of the i^{th} simplicial homology of $K_{\mathbf{b}}(M)$ with coefficients in k. To prove (i) and (ii) it suffices to show that $K_{\mathbf{b}}(\widetilde{I})$ is a cone (and therefore acyclic) unless $\mathbf{1} \preceq \mathbf{b} \preceq \mathbf{a}_I$. If $\mathbf{a}_I = (a_1, \ldots, a_n)$ and $b_i \geq a_i + 1$, then it follows from Proposition 2.6(ii) that $K_{\mathbf{b}}(\widetilde{I})$ is a cone with vertex $\{i\}$, proving (i). That $K_{\mathbf{b}}(\widetilde{I})$ is a cone with vertex $\{i\}$ if $b_i \leq 0$ follows directly from the definition of Čech hull, proving (ii). Finally, (iii) holds because $K_{\mathbf{b}}(\widetilde{I}) = K_{\mathbf{b}}(I)$ whenever $\mathbf{b} \succeq \mathbf{1}$. □

Lemma 3.8 *Let $M \subseteq T$. Then $\beta_{i,\mathbf{b}}(M) = \beta_{i,\mathbf{b}}(M/M \cap \widetilde{\mathfrak{m}^{\mathbf{a}+\mathbf{1}}})$ if $\mathbf{b} \preceq \mathbf{a}$.*

Proof: It follows from Example 2.7(ii) that $(M \cap \widetilde{\mathfrak{m}^{\mathbf{a}+\mathbf{1}}})_{\mathbf{b}} = 0$ if $\mathbf{b} \preceq \mathbf{a}$, so the Taylor resolution of it (see [16] for the original or [4], Proposition 1.5 for a treatment including submodules of T) forces $\beta_{i,\mathbf{b}}(M \cap \widetilde{\mathfrak{m}^{\mathbf{a}+\mathbf{1}}}) = 0$ for all $\mathbf{b} \preceq \mathbf{a}$. Applying $\underline{\mathrm{Tor}}$ to the exact sequence

$$0 \to M \cap \widetilde{\mathfrak{m}^{\mathbf{a}+\mathbf{1}}} \to M \to M/M \cap \widetilde{\mathfrak{m}^{\mathbf{a}+\mathbf{1}}}$$

yields the result. □

Lemma 3.9 *If $i < n$ then $\underline{\mathrm{Ext}}^i(k, S/I) \cong \underline{\mathrm{Ext}}^i(k, T/I)$, and in the remaining case $i = n$ we have $\underline{\mathrm{Ext}}^n(k, S/I) = k[\mathbf{1}]$.*

Proof: One can first calculate $\underline{\mathrm{Ext}}^i(k, S) = \begin{cases} k[\mathbf{1}] & \text{if } i = n \\ 0 & \text{otherwise} \end{cases}$ from the Koszul complex and $\underline{\mathrm{Ext}}^i(k, T) = 0$ for all i because T is injective in the category $\mathcal{M}$. Using the long exact sequence of $\underline{\mathrm{Ext}}$ from $0 \to S \to T \to T/S \to 0$ one finds that $\underline{\mathrm{Ext}}^i(k, S) \cong \underline{\mathrm{Ext}}^{i-1}(k, T/S)$.

From the above calculations and the long exact sequence of $\underline{\mathrm{Ext}}$ arising from

$$0 \to S/I \to T/I \to T/S \to 0$$

the lemma will follow if we can show that the map

$$\underline{\mathrm{Ext}}^{n-1}(k, T/S) \to \underline{\mathrm{Ext}}^{n}(k, S/I)$$

is an isomorphism. But S is a regular ring, so $\underline{\mathrm{Ext}}^{n}(k, S/I)$ is nonzero *a priori* because of [5], Proposition 3.1.14 and [5], Theorem 3.1.17, so it is enough to prove that the 1-dimensional vector space $\underline{\mathrm{Ext}}^{n-1}(k, T/S) \cong \underline{\mathrm{Ext}}^{n}(k, S) \cong k[\mathbf{1}]$ maps surjectively, i.e. that $\underline{\mathrm{Ext}}^{n}(k, T/I) = 0$. Now $\underline{\mathrm{Ext}}^{n}(k, T/I) \cong \underline{\mathrm{Ext}}^{n+1}(k, I)$ because of the exact sequence

$$0 \to I \to T \to T/I \to 0,$$

and $\underline{\mathrm{Ext}}^{n+1}(k, I) = 0$ because of the same [5] reference as above. □

The next main result, Theorem 3.12, is really a rephrasing of an observation made in the proof of [9], Theorem 5.2. While it is possible, by quoting the self-duality of the Koszul complex, to extend the result to include all S-modules, the proof here demonstrates effectively the interaction of Alexander duality with other kinds of duality. Aside from the intrinsic interest in its proof, Theorem 3.12 will find an application in the proof of Theorem 4.10. Two preliminary results are needed, the first of which will also be used in the proof of Proposition 4.6.

Lemma 3.10 *With $J = I + \mathfrak{m}^{\mathbf{a}+\mathbf{1}}$ we have $\widetilde{J}^{\,T} = I^{[\mathbf{a}]}[\mathbf{a}]$. The same is true if I and $I^{[\mathbf{a}]}$ are reversed.*

Proof: The last statement is because of Proposition 2.12. By Example 2.7(ii) and Proposition 2.4,

$$I^{[\mathbf{a}]}[\mathbf{a}] \;=\; \widetilde{I}^{\,T} \cap S[\mathbf{a}] \;=\; (\widetilde{I} + S[\mathbf{a}]^{T})^{T} \;=\; \widetilde{J}^{\,T}.$$ □

The reader knowledgeable about linkage will recognize a hint of Theorem 2.1 in the next proposition. Only the special case $\mathbf{b} = \mathbf{0}$ is required in this section. However, the more general result is a major component in the proof of Theorem 6.11.

Proposition 3.11 *Let $\mathbf{a} \succeq \mathbf{a}_I$, $J = I^{[\mathbf{a}]} + \mathfrak{m}^{\mathbf{a}+\mathbf{1}}$, and $\mathbf{b} \in \mathbb{N}^n$. Then*

$$\underline{\mathrm{Ext}}_S^{n}\Big(S[\mathbf{b}]/\,S[\mathbf{b}]\cap\widetilde{J}, S\Big) \;=\; \Big(I\,/\,I\cap\mathfrak{m}^{\mathbf{a}+\mathbf{b}+\mathbf{1}}\Big)[\mathbf{a}+\mathbf{1}].$$

In particular, taking $\mathbf{b} = \mathbf{0}$ yields $\underline{\mathrm{Ext}}^{n}(S/J, S) \cong \Big(I/\,I\cap\mathfrak{m}^{\mathbf{a}+\mathbf{1}}\Big)[\mathbf{a}+\mathbf{1}]$.

Proof: The module $T/\widetilde{J}$ is the k-dual of the finitely generated module $I[\mathbf{a}]$ by Lemma 3.10, and is hence artinian by Matlis duality, cf. [7], Theorem 2.1.4. Thus our module $S[\mathbf{b}]/\,S[\mathbf{b}]\cap\widetilde{J} \subseteq T/\widetilde{J}$ is also artinian, and (obviously) finitely generated, as well. Since the canonical module of S is $S[-\mathbf{1}]$ by [7], Corollary 2.2.6, local duality (in the form of [7], Theorem 2.2.2) applied to the zeroeth local cohomology module implies the first equality of the following:

$$\begin{aligned}
\underline{\mathrm{Ext}}^n_S\Big(S[\mathbf{b}]/\,S[\mathbf{b}]\cap\widetilde{J},S\Big) &= \Big(S[\mathbf{b}]/\,S[\mathbf{b}]\cap\widetilde{J}\Big)^\wedge[\mathbf{1}] && \text{by local duality}\\
&= \Big(\widetilde{J}^{\,T}/\,\widetilde{J}^{\,T}\cap S[\mathbf{b}]^T\Big)[\mathbf{1}] && \text{by Proposition 2.4(vii)}\\
&= \Big(I/\,I\cap S[\mathbf{a}+\mathbf{b}]^T\Big)[\mathbf{a}+\mathbf{1}] && \text{by Lemma 3.10 and shifting by } [-\mathbf{a}][\mathbf{a}]\\
&= \Big(I/\,I\cap\mathfrak{m}^{\mathbf{a}+\mathbf{b}+\mathbf{1}}\Big)[\mathbf{a}+\mathbf{1}] && \text{by Example 2.7(ii).}
\end{aligned}$$

□

Given an artinian ideal J, the list of Betti numbers for the canonical module $\underline{\mathrm{Ext}}^n(S/J,S[-\mathbf{1}])$ of S/J is essentially the reverse of the list of Betti numbers for J; see, for instance, [5], Corollary 3.3.9. On the other hand, there is the lattice-complementation view of Alexander duality, which emerges in Corollary 3.6 as a relation between the Betti numbers of a monomial module and the Bass numbers of its T-dual. These two dualities can be combined to compare the Betti numbers of I to the Bass numbers of the same ideal I:

Theorem 3.12 *For all $i\in\mathbb{Z}$ and $\mathbf{b}\in\mathbb{Z}^n$,*

$$\beta_{n-i,\,\mathbf{b}}(S/I) \;=\; \mu_{i,\,\mathbf{b}-\mathbf{1}}(S/I).$$

Proof: The case $i=n$ follows from the calculations of Lemma 3.9, so assume from now on that $i\le n-1$. In particular, we can calculate the Bass numbers from T/I instead of S/I by Lemma 3.9. Let $\mathbf{a}=\mathbf{a}_I+\mathbf{1}$. All of the Betti numbers are zero in degrees $\mathbf{b}\not\preceq\mathbf{a}$ by Proposition 3.7(i) and (iii). As for the Bass numbers, we can use the fact that, with $J:=I^{[\mathbf{a}]}+\mathfrak{m}^{\mathbf{a}+\mathbf{1}}$, we have $I^T=\widetilde{J}[\mathbf{a}]$ by Lemma 3.10. It follows that $\mu_{i,\mathbf{b}-\mathbf{1}}(T/I)=\beta_{i,\mathbf{1}-\mathbf{b}}(\widetilde{J}[\mathbf{a}])=\beta_{i,\mathbf{a}+\mathbf{1}-\mathbf{b}}(\widetilde{J})$ by Corollary 3.6, and then Proposition 3.7(ii) implies that these numbers are zero if $\mathbf{b}\not\preceq\mathbf{a}$.

From now on, assume $\mathbf{b}\preceq\mathbf{a}$ and $0\le i\le n-1$. Let $J=I^{[\mathbf{a}]}+\mathfrak{m}^{\mathbf{a}+\mathbf{1}}$ and calculate

$$\begin{aligned}
\mu_{i,\,\mathbf{b}-\mathbf{1}}(S/I) &= \mu_{i,\,\mathbf{b}-\mathbf{1}}(T/I) && \text{by Lemma 3.9 and } i\le n-1\\
&= \beta_{i,\,\mathbf{a}+\mathbf{1}-\mathbf{b}}(\widetilde{J}) && \text{by Corollary 3.6, since } I^T=\widetilde{J}[\mathbf{a}]\\
&= \beta_{i,\,\mathbf{a}+\mathbf{1}-\mathbf{b}}(J) && \text{by Proposition 3.7(iii) and } \mathbf{b}\preceq\mathbf{a}\\
&= \beta_{i+1,\,\mathbf{a}+\mathbf{1}-\mathbf{b}}(S/J) && \text{since } i\ge 0\\
&= \beta_{n-i-1,\,\mathbf{b}-\mathbf{1}-\mathbf{a}}\Big(\underline{\mathrm{Ext}}^n(S/J,S)\Big) && \text{since } S \text{ is Gorenstein and } S/J \text{ is artinian}\\
&= \beta_{n-i-1,\,\mathbf{b}}\Big((I/I\cap\mathfrak{m}^{\mathbf{a}+\mathbf{1}})\Big) && \text{by Proposition 3.11}\\
&= \beta_{n-i-1,\,\mathbf{b}}(I) && \text{by Lemma 3.8 and } \mathbf{b}\preceq\mathbf{a}\\
&= \beta_{n-i,\,\mathbf{b}}(S/I) && \text{since } i\le n-1
\end{aligned}$$

proving the theorem. □

4 Localization and restriction

This section aims to reveal the equality (Theorem 4.10) between Betti numbers of I and localized Bass numbers (Definition 4.8) of $I^{[\mathbf{a}]}$. This equality generalizes Theorem 2.13. As a consequence of the equality, an inequality between Betti numbers of I and $I^{[\mathbf{a}]}$ is obtained in Theorem 4.13, generalizing to arbitrary monomial ideals an inequality of [2] which was proven for radical ideals.

The next proposition should be thought of as the nonlocalized precursor to Theorem 4.10(i).

Proposition 4.1 *Let I be an ideal and $\mathbf{a} \succeq \mathbf{a}_I$. If $\mathbf{1} \preceq \mathbf{b} \preceq \mathbf{a}$ then $\beta_{i,\mathbf{b}}(\, I\,) = \mu_{i,\mathbf{b}^{\mathbf{a}}-\mathbf{1}}(\, S/I^{[\mathbf{a}]}\,)$.*

Proof: If, to start with, $\mathbf{b} \preceq \mathbf{a}$, then

$$\begin{aligned} \beta_{i\,,\,\mathbf{b}}(\,M\,) &= \mu_{i\,,\,-\mathbf{b}}\Big((M/M \cap \widetilde{\mathfrak{m}^{\mathbf{a}+\mathbf{1}}})^\wedge\Big) && \text{by Lemma 3.8 and Proposition 3.5} \\ &= \mu_{i\,,\,-\mathbf{b}}\Big(S[\mathbf{a}]/S[\mathbf{a}] \cap M^T\Big) && \text{by Proposition 2.4(vii) and Example 2.7(ii)} \\ &= \mu_{i\,,\,\mathbf{a}-\mathbf{b}}\Big(S/M^T[-\mathbf{a}] \cap S\Big). \end{aligned}$$

Substituting $M = \widetilde{I}$ we get $\beta_{i\,,\,\mathbf{b}}(\,\widetilde{I}\,) = \mu_{i\,,\,\mathbf{a}-\mathbf{b}}(\,S/I^{[\mathbf{a}]}\,)$ if $\mathbf{b} \preceq \mathbf{a}$, and when the assumption $\mathbf{1} \preceq \mathbf{b}$ is added, $\mathbf{a} - \mathbf{b} = \mathbf{b}^{\mathbf{a}} - \mathbf{1}$ and the result is a consequence of Proposition 3.7(iii). □

Theorem 4.10 is the combination of the previous proposition with localization and restriction of scalars. The following definitions will provide concise notation for these operations, which will be needed also for the definition of Bass numbers at primes other than $\mathfrak{m}$ (Definition 4.8). Recall that $\overline{F} = \{1, \ldots, n\} \setminus F = \mathbf{1} - F$.

Definition 4.2 *Let Δ be the $(n-1)$-simplex on the vertices $\{1, \ldots, n\}$ and $F \in \Delta$. Define*

(i)	$N(-\overline{F}\,)$	$:=$	$S[x^{-\overline{F}}\,] \otimes_S N$	*for arbitrary modules N*
(ii)	$S_{[F]}$	$:=$	$k[x_i \mid i \in F]$	*a $\mathbb{Z}^F$-graded k-subalgebra of S*
(iii)	$N_{[F]}$	$:=$	$\bigoplus_{\mathbf{b}\in\mathbb{Z}^F} N_{\mathbf{b}}$	*a $\mathbb{Z}^F$-graded $S_{[F]}$-module*
(iv)	$N_{(F)}$	$:=$	$N(-\overline{F}\,)_{[F]}$	

The operations on N listed above are all exact and commute with sums. They should be thought of as: (i) homogeneous localization at $\mathfrak{m}^F$, (iii) taking the "degree zero part" of N with respect to F, and (iv) taking the "degree zero part of the homogeneous localization at $\mathfrak{m}^F$" as in algebraic geometry. In (ii) and (iii), the copy of $\mathbb{Z}^F$ may be thought of as sitting inside $\mathbb{Z}^n$ in the obvious way: as the space spanned by the basis vectors in the support of F. Thinking of $\mathbb{Z}^F$ this way can cause notational problems, however. For instance, any $\mathbb{Z}^n$-graded S-module N can be thought of as a $\mathbb{Z}^F$-graded $S_{[F]}$-module which in degree $\mathbf{b} \in \mathbb{Z}^F$ is

$$\bigoplus_{\mathbf{c}\cdot F\,=\,\mathbf{b}} N_{\mathbf{c}} \;=\; \bigoplus_{\mathbf{b}'\,\in\,\mathbb{Z}^{\overline{F}}} N_{\mathbf{b}\,+\,\mathbf{b}'}\,,$$

where $\mathbf{c} \cdot F$ denotes the restriction to F as in Section 1. Note that the right-hand side gives this vector space the structure of a $\mathbb{Z}^{\overline{F}}$-graded $S_{[\overline{F}\,]}$-module. The convention will be the following:

> If N is a $\mathbb{Z}^F$-graded $S_{[F]}$-module and $\mathbf{b} \in \mathbb{Z}^F$, the graded piece of N in degree $\mathbf{b}$ will be denoted $N_{\mathbf{b}\cdot F}$. That way, if N happens also to be a $\mathbb{Z}^n$-graded S-module, the usual notation $N_{\mathbf{b}}$ can retain its old meaning as the degree $\mathbf{b}$ part in the $\mathbb{Z}^n$-grading.

Even if $\mathbf{b} \notin \mathbb{Z}^F$ it will sometimes be convenient to use $N_{\mathbf{b}\cdot F}$ to denote the $\mathbf{b} \cdot F$ graded piece in the $\mathbb{Z}^F$-grading; i.e. with $\mathbf{c} = \mathbf{b} \cdot F \in \mathbb{Z}^F$, we set $N_{\mathbf{b}\cdot F} := N_{\mathbf{c}\cdot F}$. The next Lemma follows from the definitions and the convention above. In each of (i)–(v), the objects are $\mathbb{Z}^F$-graded $S_{[F\,]}$-modules, but in (i), the objects may also be considered as $\mathbb{Z}^{\overline{F}}$-graded $S_{[\overline{F}\,]}$-modules or even $\mathbb{Z}^{\overline{F}} \times \mathbb{Z}^F = \mathbb{Z}^n$-graded $S_{[\overline{F}\,]} \otimes_k S_{[F\,]} = S$-modules.

Lemma 4.3 *For any $F \in \Delta$,*

$$
\begin{array}{lrcl}
(i) & M(-\overline{F}\,) & = & T_{[\overline{F}]} \otimes_k M_{(F)} \;=\; S(-\overline{F}\,) \otimes_{S_{(F)}} M_{(F)} \\
(ii) & M_{[F]} & = & M_{\mathbf{0}\cdot\overline{F}} \\
(iii) & M[\mathbf{a}]_{[F]} & = & M_{\mathbf{a}\cdot\overline{F}}[\mathbf{a} \cdot F] \\
(iv) & (\widetilde{I}\,)_{[F]} & = & \widetilde{I_{[F]}} \\
(v) & (M^T)_{[F]} & = & (M_{[F]})^{T_{[F]}}
\end{array}
$$

where the right-hand sides of (iv) and (v) are, respectively, the Čech hull and T-dual over $S_{[F]}$. $\square$

For submodules $M \subseteq T$ the various gradings allow for convenient characterizations of localization as in Definition 4.2(iv). They use the fact that for any $\mathbf{b} \in \mathbb{Z}^n$, $M_{\mathbf{b}\cdot\overline{F}}$ is naturally a submodule of $T_{[F]} = T_{(F)}$.

Proposition 4.4 *Let M be a monomial module.*

$$(i) \quad M_{(F)} \;=\; \bigcup_{\mathbf{b}\,\in\,\mathbb{Z}^{\overline{F}}} M_{\mathbf{b}\cdot\overline{F}}\,.$$

If M can be generated in degrees $\mathbf{c}$ satisfying $\mathbf{c} \cdot \overline{F} \preceq \mathbf{a} \cdot \overline{F}$ then

$$(ii) \quad M_{(F)} \;=\; M_{\mathbf{a}\cdot\overline{F}}\,.$$

Proof: (i) Observe that $M \subseteq M(-\overline{F}\,) \subseteq T$ because everything is torsion-free. Thus, if $\mathbf{b} \in \mathbb{Z}^{\overline{F}}$, then multiplication by $x^{-\mathbf{b}}$ induces an inclusion $M_{\mathbf{b}\cdot\overline{F}} \to M_{(F)}$. For the other inclusion, note that any monomial in $M_{(F)}$ can be written as $x^{\mathbf{b}} \cdot x^{\mathbf{c}}$ for some $x^{\mathbf{c}} \in M$ and $\mathbf{b} = -(\mathbf{c} \cdot \overline{F}) \in \mathbb{Z}^{\overline{F}}$.

(ii) The collection $\{M_{\mathbf{b}\cdot\overline{F}}\}_{\mathbf{b}\,\in\,\mathbb{Z}^{\overline{F}}}$ of $S_{[F]}$-submodules of $T_{[F]}$ is partially ordered by inclusion because M is a module. The union in (i) stabilizes after $\mathbf{a} \cdot \overline{F}$ if M is generated in degrees $\mathbf{c}$ satisfying $\mathbf{c} \cdot \overline{F} \preceq \mathbf{a} \cdot \overline{F}$. $\square$

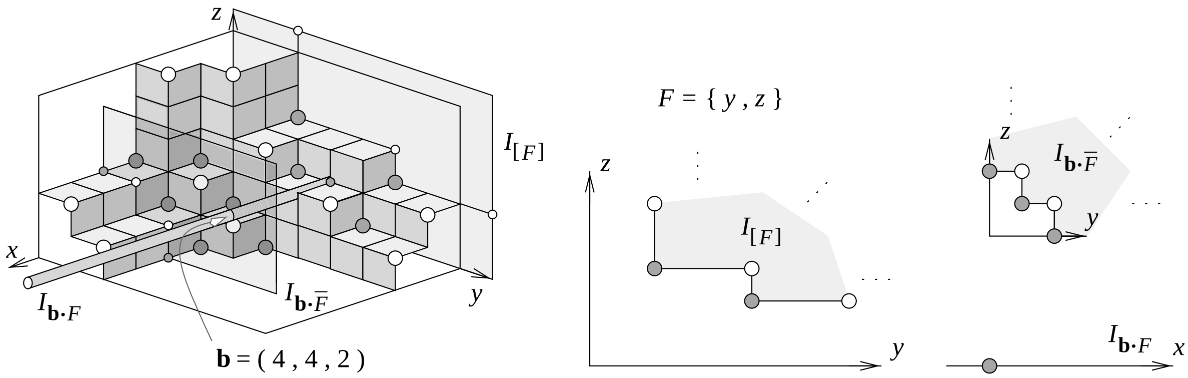


Figure 3

Example 4.5 Figure 3 illustrates some parts of Definition 4.2 and Lemma 4.3 in a specific case. For notation, x, y, and z are identified with 1, 2, and $3 \in \{1, \ldots, 3\} = \Delta$. The face F is $\{y, z\} = \{2, 3\}$, while $\mathbf{b} = (4, 4, 2)$. The small colored dots represent generators or irreducible components in the restricted ideals. It is not true that $\mathbf{b} \succeq \mathbf{a}_I$, so Proposition 4.4 does not apply; nevertheless, $I_{\mathbf{b}\cdot\overline{F}} = I_{(F)}$ for these $\mathbf{b}$, I, and F. Figure 3 can also be used as a test case for Proposition 4.6.

Proposition 4.6 $(I^{[\mathbf{a}]})_{(F)} = (I_{[F]})^{[\mathbf{a}\cdot F]}$ *as ideals in* $S_{(F)} = S_{[F]}$. *In words, dualizing and then localizing is the same as restricting and then dualizing.*

Proof: It is enough to show that $(I^{[\mathbf{a}]})_{(F)}[\mathbf{a} \cdot F] = (I_{[F]})^{[\mathbf{a}\cdot F]}[\mathbf{a} \cdot F]$. Now

$$\begin{aligned}
(I^{[\mathbf{a}]})_{(F)}[\mathbf{a} \cdot F] &= (I^{[\mathbf{a}]})_{\mathbf{a}\cdot\overline{F}}[\mathbf{a} \cdot F] && \text{by Proposition 4.4(ii) and Proposition 2.12} \\
&= (I^{[\mathbf{a}]}[\mathbf{a}])_{[F]} && \text{by Lemma 4.3(iii)} \\
&= \left(\left((I + \mathfrak{m}^{\mathbf{a}+\mathbf{1}})^{\widetilde{}}\right)^{T}\right)_{[F]} && \text{by Lemma 3.10,}
\end{aligned}$$

and one can use the rules 4.3(v) and then 4.3(iv) for interchanging the various operations to get the last line to equal

$$\left((I_{[F]} + \mathfrak{m}^{\mathbf{a}\cdot F + F}{}_{[F]})^{\widetilde{}}\right)^{T_{[F]}},$$

where $(-)^{T_{[F]}}$ is T-duality over $S_{[F]}$ as in Lemma 4.3(v). Another application of Lemma 3.10 (over $S_{[F]}$ this time) gives the desired result. $\square$

Proposition 4.7 *Let* $I \subseteq S$ *and* $\mathbf{b} \in \mathbb{Z}^F$. *Then* $\beta_{i,\mathbf{b}}(I) = \beta_{i,\mathbf{b}\cdot F}(I_{[F]})$.

Proof: Let $\mathbb{F}$ be the Taylor resolution of I (see the proof of Lemma 3.8 for references). Then $\mathbb{F}_{[F]}$ is the Taylor resolution of $I_{[F]}$. Furthermore, $(k \otimes_S \mathbb{F})_{[F]} = k \otimes_{S_{[F]}} \mathbb{F}_{[F]}$ because if $\mathbf{b} \in \mathbb{N}^n$ then

$$\Big(k \otimes_S S[-\mathbf{b}]\Big)_{[F]} = k \otimes_{S_{[F]}} S[-\mathbf{b}]_{[F]} = \begin{cases} k[-\mathbf{b}] & \text{if } \mathbf{b} \in \mathbb{Z}^F \\ 0 & \text{if } \mathbf{b} \notin \mathbb{Z}^F \end{cases}.$$

Thus the Betti numbers in question are calculated from the same complex of k-vector spaces. $\square$

Definition 4.8 (Bass numbers for arbitrary monomial primes) *Given a module N and a degree $\mathbf{b} \in \mathbb{Z}^F$, the i^{th}* Bass number of N with respect to F (or the prime ideal $\mathfrak{m}^F$) in degree $\mathbf{b}$ *is defined as*

$$\mu_{i,\mathbf{b}}(F,N) \;:=\; \dim_k\Big(\underline{\mathrm{Ext}}^{\,i}_{S_{(F)}}(k, N_{(F)})_{\mathbf{b}}\Big)\,.$$

Remark 4.9 When $F = \mathbf{1}$ this definition agrees with the Bass numbers of Definition 3.1.

Now comes the main result of this section. It can be thought of as a far-reaching generalization of Theorem 2.13, which is a special case, pending the appropriate interpretation of Bass numbers—see Proposition 4.12 and the second proof of Theorem 2.13 that follows it. In part (i) of the next theorem, the case where $\mathbf{b}$ has full support is Proposition 4.1.

Theorem 4.10 *If $\mathbf{0} \neq F \preceq \mathbf{b} \preceq \mathbf{a}{\cdot}F$ then for all $i \in \mathbb{Z}$ we have*

$$\begin{array}{lrcl} (i) & \beta_{i\,,\mathbf{b}}(I) & = & \mu_{i\,,\mathbf{b}^{\mathbf{a}}-F}(F, S/I^{[\mathbf{a}]}) \\ (ii) & \mu_{n-i-1\,,\mathbf{b}-\mathbf{1}}(S/I) & = & \mu_{i\,,\mathbf{b}^{\mathbf{a}}-F}(F, S/I^{[\mathbf{a}]}) \\ (iii) & \beta_{i\,,\mathbf{b}}(I) & = & \beta_{|F|-i-1\,,\mathbf{b}^{\mathbf{a}}}({I^{[\mathbf{a}]}}_{(F)}). \end{array}$$

In any of these formulas, I and $I^{[\mathbf{a}]}$ can be switched, and the same goes for $\mathbf{b}$ and $\mathbf{b}^{\mathbf{a}}$.

Proof: Statements (ii) and (iii) follow easily from (i), in view of Theorem 3.12 and the fact that $\beta_{i,\mathbf{b}}(I) = \beta_{i+1,\mathbf{b}}(S/I)$ when $\mathbf{b} \neq \mathbf{0}$. To prove (i), note that $\mathbf{b}^{\mathbf{a}} = (\mathbf{b} \cdot F)^{\mathbf{a}\cdot F}$, so

$$\begin{array}{rcll} \beta_{i\,,\mathbf{b}}(\,I\,) & = & \beta_{i\,,\mathbf{b}\cdot F}(\,I_{[F]}\,) & \text{by Proposition 4.7} \\ & = & \mu_{i\,,\mathbf{b}^{\mathbf{a}}-F}(\,S_{[F]}/{I_{[F]}}^{[\mathbf{a}\cdot F]}\,) & \text{by Proposition 4.1} \\ & = & \mu_{i\,,\mathbf{b}^{\mathbf{a}}-F}(\,S_{(F)}/{I^{[\mathbf{a}]}}_{(F)}\,) & \text{by Proposition 4.6} \\ & = & \mu_{i\,,\mathbf{b}^{\mathbf{a}}-F}(\,F\,,\,S/I^{[\mathbf{a}]}\,) & \text{by definition} \end{array}$$

since $(-)_{(F)}$ is exact. Note that the Bass number in the penultimate line is with respect to the maximal ideal of $S_{(F)}$. The last statement of the theorem is true because $(\mathbf{b}^{\mathbf{a}})^{\mathbf{a}} = \mathbf{b}$ and $(I^{[\mathbf{a}]})^{[\mathbf{a}]} = I$, and because imposing the condition on $\mathbf{b}$ is equivalent to imposing the same condition on $\mathbf{b}^{\mathbf{a}}$. $\square$

Remark 4.11 Part (i) of the theorem can be thought of as the generalization to arbitrary monomial ideals of the formulas in [6], Proposition 1 and [2], Theorem 2.4, using [9], Theorem 5.2 and the fact that links come from localization ([9], Proposition 5.6).

As a consequence of the theorem, the list of Betti numbers of $I^{\mathbf{a}}$ will be independent of $\mathbf{a}$, though the $\mathbb{Z}^n$-degrees in which they occur will vary with $\mathbf{a}$. Indeed, the list of Betti numbers of $I^{\mathbf{a}}$ is just the list of (localized) Bass numbers of I by part (i) of the theorem. Thus the collection of ideals that are dual to I are very closely related homologically. This will be highlighted again in Section 5 in terms of various geometrically defined resolutions.

Before the above remark, the symbol $I^{\mathbf{a}}$ had not appeared in this section (or the last) without brackets on the $\mathbf{a}$; that is, none of the results have been logically dependent on Definition 1.5 or Theorem 2.13. Therefore, Theorem 4.10 can be used to give a second proof of Theorem 2.13. In fact, this "second proof" was discovered before the more elementary proof in Section 2. The next proposition is what allows the irreducible decomposition to be read off of the zeroeth Bass numbers just as the minimal generators are read off the zeroeth Betti numbers.

Proposition 4.12 *Given an ideal $I \subseteq S$ the following are equivalent for $\mathbf{b} \in \mathbb{Z}^F$:*

(i) $\mathfrak{m}^{\mathbf{b}}$ is an irredundant irreducible component of I.
(ii) $\mu_{0,\mathbf{b}-F}(F, S/I) = 1$.
(iii) $\mu_{0,\mathbf{b}-F}(F, S/I) \neq 0$.

Proof: Let $I = \bigcap_j Q_j$ be the (unique) irredundant decomposition of I into irreducible ideals Q_j. Then we have an injection $0 \to S/I \to \bigoplus_j S/Q_j$ which, by the proofs of [17], Propositions 3.16 and 3.17, induces an *isomorphism*

$$\underline{\mathrm{Hom}}_S(S/\mathfrak{m}^F, S/I)(-\overline{F}\,) \to \bigoplus_j \underline{\mathrm{Hom}}_S(S/\mathfrak{m}^F, S/Q_j)(-\overline{F}\,); \tag{3}$$

this is because the functor $\Delta_{\mathfrak{p}}(\cdot)_{\mathfrak{p}}$ in the [17] reference is easily seen to be $\mathrm{Hom}\,(R/\mathfrak{p}, \cdot)_{\mathfrak{p}}$ (so we can take $\mathfrak{p} = \mathfrak{m}^F$). Using Lemma 4.3(i) we can move the localization into and out of the $\underline{\mathrm{Hom}}$: for any finitely generated S-modules L and N,

$$\begin{aligned} \underline{\mathrm{Hom}}_S\Big(L, N\Big)(-\overline{F}) &\cong \underline{\mathrm{Hom}}_{S(-\overline{F})}\Big(L(-\overline{F}), N(-\overline{F})\Big) \\ &\cong S(-\overline{F}) \otimes_{S_{(F)}} \underline{\mathrm{Hom}}_{S_{(F)}}\Big(L_{(F)}, N_{(F)}\Big) \\ &\cong T_{[\overline{F}]} \otimes_k \underline{\mathrm{Hom}}_{S_{(F)}}\Big(L_{(F)}, N_{(F)}\Big)\,. \end{aligned}$$

Treating these as $\mathbb{Z}^{\overline{F}}$-graded $S_{[\overline{F}]}$-modules and taking the degree $\mathbf{0}\cdot\overline{F}$ part in the last line yields $\underline{\mathrm{Hom}}_{S_{(F)}}\Big(L_{(F)}, N_{(F)}\Big)$. Applying this procedure to Equation (3) reveals an isomorphism

$$\underline{\mathrm{Hom}}_{S_{(F)}}\Big(k, (S/I)_{(F)}\Big) \;\cong\; \bigoplus_j \underline{\mathrm{Hom}}_{S_{(F)}}\Big(k, (S/Q_j)_{(F)}\Big).$$

Since we can calculate

$$\mu_{\,0,\mathbf{b}-F}\,(F, S/Q_j) \;=\; \begin{cases} 1 & \text{if } Q_j = \mathfrak{m}^{\mathbf{b}} \\ 0 & \text{otherwise} \end{cases}$$

the proposition follows from the definition of Bass numbers. □

Second proof of Theorem 2.13: Every generator $x^{\mathbf{b}}$ of I corresponds to a nontrivial zeroeth Betti number of I which satisfies the condition $F \preceq \mathbf{b} \preceq \mathbf{a} \cdot F$ for $F = \sqrt{\mathbf{b}}$ because $I \subseteq S$ and $\mathbf{a} \succeq \mathbf{a}_I$. After applying Theorem 4.10(i) and the previous proposition, we can conclude that each generator of I does indicate the presence of an appropriate irreducible component of $I^{[\mathbf{a}]}$. To show that each nontrivial zeroeth Bass number of $I^{[\mathbf{a}]}$ comes from some Betti number of I, we demonstrate that if $\mathbf{b} \in \mathbb{Z}^F$ and $\mu_{0,\mathbf{b}-F}(F, S/I) \neq 0$ then $F \preceq \mathbf{b} \preceq \mathbf{a} \cdot F$. Localizing at $\mathfrak{m}^F$, we may assume that $F = \mathbf{1}$. Clearly $\mathbf{b} \succeq \mathbf{1}$ since $\mathfrak{m}^{\mathbf{b}}$ is $\mathfrak{m}$-primary, so the desired result falls out of Theorem 3.12 and Proposition 3.7. □

Next on the agenda is the generalization to arbitrary monomial ideals of an inequality of [2] for squarefree ideals. The topological argument involving links employed there is preempted here by a simple algebraic observation involving localization (which gives links in the squarefree case, see [9], Proposition 5.6).

Theorem 4.13 *If* $\mathbf{a} \succeq \mathbf{a}_I$ *and* $F \preceq \mathbf{b} \preceq \mathbf{a} \cdot F$ *then*

$$\beta_{i,\mathbf{b}}(I) \quad \leq \sum_{\mathbf{c}\cdot F = \mathbf{b}^{\mathbf{a}}} \beta_{|F|-i-1,\mathbf{c}}(I^{\mathbf{a}}).$$

Proof: Let $\mathbb{F}$ be a minimal free resolution of $I^{\mathbf{a}}$. Localizing at $\mathfrak{m}^F$ we obtain a free resolution $\mathbb{F}_{(F)}$ of ${I^{\mathbf{a}}}_{(F)}$ over $S_{(F)}$. The generators of $\mathbb{F}_{(F)}$ as a free $S_{(F)}$-module are in bijective correspondence with the generators of $\mathbb{F}$ itself. Specifically, for any $\mathbf{b}' \in \mathbb{Z}^F$ we find that $S[\mathbf{c}]_{(F)} = S_{(F)}[\mathbf{b}' \cdot F]$ if and only if $\mathbf{c} \cdot F = \mathbf{b}'$. Thus the number of summands of $\mathbb{F}_{(F)}$ in homological degree $|F| - i - 1$ and $\mathbb{Z}^F$-degree $\mathbf{b}^{\mathbf{a}}$ is equal to

$$\sum_{\mathbf{c}\cdot F = \mathbf{b}^{\mathbf{a}}} \beta_{|F|-i-1,\mathbf{c}}(I^{\mathbf{a}})$$

since $\mathbb{F}$ is minimal. On the other hand, the number of such summands is clearly $\geq \beta_{|F|-i-1,\mathbf{b}^{\mathbf{a}}}({I^{\mathbf{a}}}_{(F)})$, with equality if and only if $\mathbb{F}_{(F)}$ is minimal. Since this last number is equal to $\beta_{i,\mathbf{b}}(I)$ by Theorem 4.10, we are done. □

Corollary 4.14 (Bayer-Charalambous-Popescu) *If* I *is squarefree then*

$$\beta_{i,\mathbf{b}}(I) \quad \leq \sum_{\mathbf{b}\preceq\mathbf{c}\preceq\mathbf{1}} \beta_{|\mathbf{b}|-i-1,\mathbf{c}}(I^{\vee})$$

for $0 \leq i \leq n-1$ *and* $\mathbf{0} \preceq \mathbf{b} \preceq \mathbf{1}$.

Proof: This is a special case of the theorem once it is noted that (i) $\beta_{|\mathbf{b}|-i-1,\mathbf{c}}(I^{\vee}) = 0$ unless $\mathbf{0} \preceq \mathbf{c} \preceq \mathbf{1}$, and (ii) $\mathbf{0} \preceq \mathbf{c}$ and $\mathbf{c} \cdot \sqrt{\mathbf{b}} = \mathbf{b}$ imply $\mathbf{c} \succeq \mathbf{b}$. □

5 Duality for cellular complexes: the cohull resolution

This section explores the effect of Alexander duality on various geometrically defined free resolutions, in the spirit of [3], [4], and [15]. First, the concept of a geometrically defined resolution is broadened past *cellular resolutions* to include *relative cocellular resolutions* (Definition 5.3). The key result (Theorem 5.8) is presented, though the majority of its proof occupies Section 6. As an application, it is shown how irreducible decompositions can be specified by cellular resolutions (Theorem 5.12). The culmination of these ideas is a new canonical geometric resolution for monomial ideals (Definition 5.15). It is called the *cohull resolution*, and is defined by applying Alexander duality to the hull resolution of [4]. As a special case, the co-Scarf resolution of a cogeneric monomial ideal of [15] is seen to be the cohull resolution (Theorem 5.23), and is thus Alexander dual to the Scarf resolution of a generic monomial ideal [3]. A number of examples are presented, including permutahedron and forest ideals.

Conventions regarding grading and chain complexes:

A chain complex of S-modules

$$\mathbb{F}: \quad \cdots \to N_{i+1} \to N_i \to N_{i-1} \to \cdots, \quad N_i \text{ in homological degree } i,$$

is viewed as a (homologically) $\mathbb{Z}$-graded S-module $\bigoplus N_i$ with a differential ∂ of degree -1. If "$[\mathbf{a}]$" is attached to $\mathbb{F}$ then each summand is to be shifted in its $\mathbb{Z}^n$-grading to the left by $\mathbf{a}$, while "(j)" indicates that the homological degrees are to be shifted down by j, yielding the notation

$$\mathbb{F}[\mathbf{a}](j): \quad \cdots \to N_{i+1}[\mathbf{a}] \to N_i[\mathbf{a}] \to N_{i-1}[\mathbf{a}] \to \cdots, \quad N_i \text{ in homological degree } i-j.$$

Here, $N[\mathbf{a}]_{\mathbf{b}} = N_{\mathbf{a}+\mathbf{b}}$ for any S-module N by definition. Taking the S-dual $\mathbb{F}^* := \underline{\mathrm{Hom}}(\mathbb{F}, S)$ changes ∂ to its transpose δ, and makes homological degrees into cohomological degrees, which are the negatives of homological degrees:

$$\mathbb{F}^*: \quad \cdots \leftarrow N_{i+1}^* \leftarrow N_i^* \leftarrow N_{i-1}^* \leftarrow \cdots, \quad \begin{aligned} N_i^* &\text{ in homological degree } -i \\ &= \text{ cohomological degree } i. \end{aligned}$$

Labelled cell complexes provide compact vessels for recording the monomial entries in certain $\mathbb{Z}^n$-graded free resolutions of an ideal. [4] introduces this notion in the context of monomial modules, but attention is restricted to boundary operators of the cell complex. The definitions below extend the concept to include coboundary operators, as well. For the reader's convenience, the definition of a labelled regular cell complex and the cellular free complex it determines is recalled briefly below, although the reader is urged to consult [4], Section 1 for more details.

Let $\Lambda \subseteq \mathbb{Z}^n$ be a set of vectors, and let X be a *regular cell complex* whose vertices are indexed by the elements of Λ. For $\mathbf{c}, \mathbf{c}' \in \mathbb{Z}^n$, define the *join* $\mathbf{c} \vee \mathbf{c}'$ to be the componentwise maximum, i.e. $\mathbf{c} \vee \mathbf{c}'$ is the smallest vector which is greater than or equal to both $\mathbf{c}$ and $\mathbf{c}'$ in each coordinate: $(\mathbf{c} \vee \mathbf{c}')_i = \max(c_i, c_i')$. Given a face $F \in X$, define the *label* $\mathbf{a}_F$ *of* F to be the join $\bigvee_{v \in F} \mathbf{a}_v$ of the labels on the vertices in F, where $\mathbf{a}_v \in \Lambda$ is the element corresponding to v. To avoid confusion, the symbol $|X|$ will be used to denote the unlabelled underlying cell complex of the labelled cell complex X.

We assume that $|X|$ comes equipped with an *incidence function* $\varepsilon(F, F') \in \{1, 0, -1\}$ defined on pairs of faces, which is used to define the boundary map in the oriented chain complex of $|X|$ (with coefficients in k). For each $F \in X$, let SF be the free S-module with one generator F in degree $\mathbf{a}_F$. The *cellular complex* $\mathbb{F}_X$ is the homologically and $\mathbb{Z}^n$-graded chain complex of S-modules

$$\mathbb{F}_X \;=\; \bigoplus_{F \in X,\, F \neq \emptyset} SF \qquad \text{with differential} \qquad \partial F \;=\; \sum_{G \in X,\, G \neq \emptyset} \varepsilon(F, G) \frac{m_F}{m_G} \cdot G,$$

where $m_F := x^{\mathbf{a}_F}$. The homological degree of the basis vector $F \in SF$ is the dimension of $F \in |X|$. If $\mathbb{F}_X$ is acyclic, it will be said that X *supports a free resolution* of the module $\langle x^{\mathbf{a}_v} \mid v \in X$ is a vertex $\rangle$.

Remark 5.1 In [4] it is assumed that the elements of Λ are pairwise incomparable (as elements in the poset $\mathbb{Z}^n$), but Λ is not assumed to be finite. Here, however, Λ will always be finite, but pairwise incomparability is not assumed. It is easily verified that all of the results in [4], Section 1 remain true under these hypotheses.

Definition 5.2 (Relative Cellular Complexes) *A* relative cellular complex $\mathbb{F}_{(X,Y)}$ *is the quotient of a cellular complex* $\mathbb{F}_X$ *supported on a labelled regular cell complex* X *by a subcomplex* $\mathbb{F}_Y$ *for some regular cell subcomplex* $Y \subseteq X$*, with the labelling on* Y *induced by the labelling on* X*.*

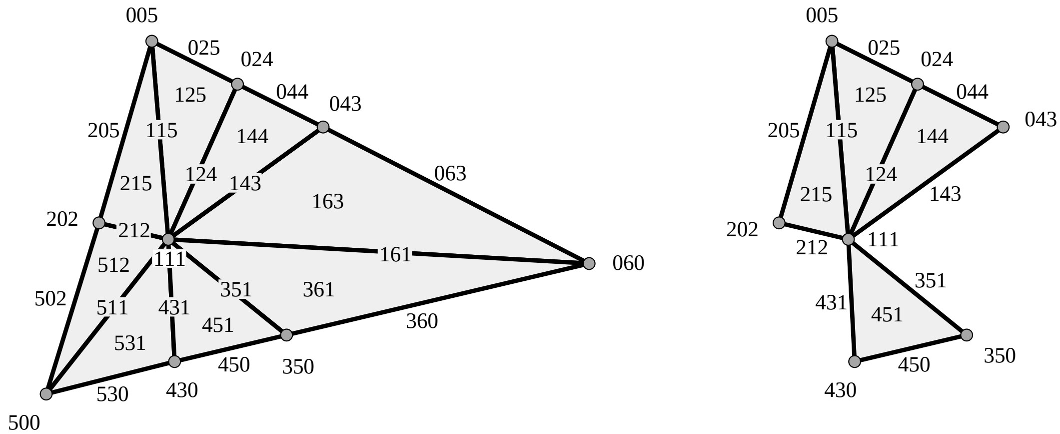
Definition 5.3 (Relative Cocellular Complexes) *A* relative cocellular complex $\mathbb{F}^{(X,Y)}$ *is obtained by taking* $\mathbb{F}^*_{(X,Y)}$ *for a pair* (X, Y) *of labelled relative regular cell complexes. If* Y *is empty,* $\mathbb{F}^{(X,Y)}$ *may be denoted* $\mathbb{F}^X$ *and called a* cocellular complex *supported on* X*.*

Remark 5.4 The relative cocellular complex $\mathbb{F}^{(X,Y)}$ can be viewed as the homogenization of the relative cochain complex of the pair (X, Y), as long as the label on a dual face F^* is the negative $-\mathbf{a}_F$ of the label on the face F. The coboundary can then be written as $\delta G^* = \sum_{(F \in X,\, F \neq \emptyset)} \varepsilon(F, G) \frac{m_F}{m_G} \cdot F^*$.

Definition 5.5 *Given a labelled regular cell complex* X *and a vector* $\mathbf{b} \in \mathbb{Z}^n$*, define the following two labelled subcomplexes of* X*:*

(*i*) $X_B(\mathbf{b}) := \{F \in X \mid \mathbf{a}_F \preceq \mathbf{b}\}$, the positively bounded subcomplex of X with respect to $\mathbf{b}$.
(*ii*) $X_U(\mathbf{b}) := \{F \in X \mid \mathbf{a}_F \not\preceq \mathbf{b}\}$, the negatively unbounded subcomplex of X with respect to $\mathbf{b}$.

Finally, let $X_U := X_U(\mathbf{1})$ *be simply* the negatively unbounded subcomplex of X.



X $\qquad$ $X_B(4,5,5)$

Figure 4

Example 5.6 Let I be as in Example 1.8. The labelled complex X in Figure 4 is the *Scarf complex* [3] of $I + \mathfrak{m}^{(5,6,6)}$ (see also Example 5.14, below). Hence $\mathbb{F}_X$ is a minimal free resolution by [3], Theorem 3.2. In this case, $(5,6,6) = \mathbf{a}_I + \mathbf{1}$, but z^5 is already in I. The label "215" in the diagrams is short for (2,1,5). The subcomplex $X_B(4,5,5)$, which is the Scarf complex of the ideal I itself, is also depicted in Figure 4 (see Proposition 5.7, below). The subcomplex X_U is depicted in Figure 5 along with a representation of the labelled relative cellular complex (X, X_U) and the relative cocellular complex $\mathbb{F}^{(X,X_U)}$ of free S-modules determined by it. For this, the edges have been oriented towards the center and the faces counterclockwise. The left copy of S^8 represents the 2-cells in clockwise order starting from 361, the right copy of S^8 represents the edges clockwise starting from 161, and the copy of S represents the lone vertex. The other vertices and edges are not considered since they lie in the subcomplex X_U. It is not a coincidence that the negatively unbounded subcomplex of X is the topological boundary of X—this will always happen for the Scarf complex of a generic artinian monomial ideal, cf. Theorem 5.18. $\square$

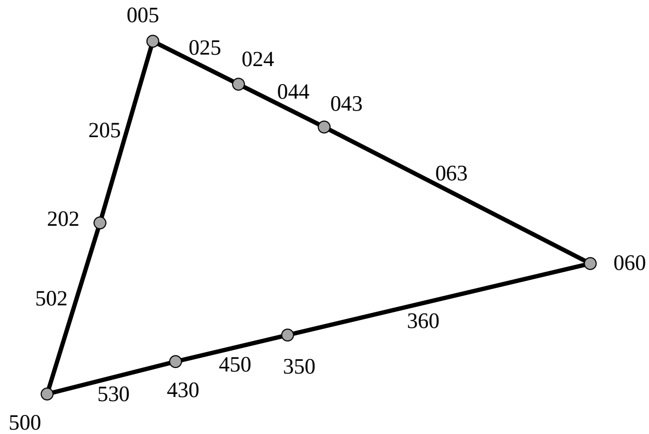
Recall that $\mathbf{a}_I$ is the exponent on the least common multiple of the minimal generators for I. Suppose that we have a cellular resolution $\mathbb{F}_X$ of the ideal $I + \mathfrak{m}^{\mathbf{a}+\mathbf{1}}$ with $\mathbf{a} \succeq \mathbf{a}_I$.

Proposition 5.7 *$\mathbb{F}_{X_B(\mathbf{b})}$ is a cellular resolution of I for any $\mathbf{b}$ such that $\mathbf{a}_I \preceq \mathbf{b} \preceq \mathbf{a}$.*

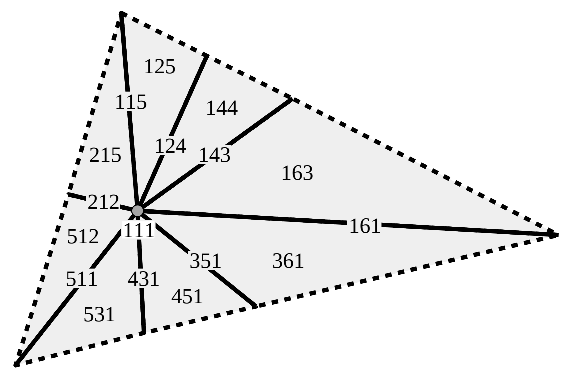
Proof: With the conditions on $\mathbf{b}$, the ideal I is generated by all monomials in $I + \mathfrak{m}^{\mathbf{a}+\mathbf{1}}$ whose exponent is $\preceq \mathbf{b}$, so the result is a direct consequence of [4], Corollary 1.3. $\square$

Duality for cellular resolutions says that if the cellular resolution $\mathbb{F}_X$ of $I + \mathfrak{m}^{\mathbf{a}+\mathbf{1}}$ has minimal length, a resolution for the Alexander dual $I^{\mathbf{a}}$ with respect to $\mathbf{a}$ can also be recovered from X:

Theorem 5.8 *If the cellular resolution $\mathbb{F}_X$ of $I + \mathfrak{m}^{\mathbf{a}+\mathbf{1}}$ has length $n-1$ then $\mathbb{F}^{(X,X_U)}[-\mathbf{a}-\mathbf{1}](1-n)$ is a relative cocellular resolution of $I^{\mathbf{a}}$. Furthermore, this dual resolution is minimal if $\mathbb{F}_X$ is.*



$X_U = X_U(\mathbf{1})$



(X, X_U)

$$0 \leftarrow S^8 \xleftarrow{\begin{pmatrix} x^2 & -y & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x & -y^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x & -y^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z & -x^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & z^3 & -x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & y & -z & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & y^2 & -z \\ -z^2 & 0 & 0 & 0 & 0 & 0 & 0 & y^2 \end{pmatrix}} S^8 \xleftarrow{\begin{pmatrix} y^5 \\ x^2y^4 \\ x^3y^2 \\ x^4 \\ xz \\ z^4 \\ yz^3 \\ y^3z^2 \end{pmatrix}} S \leftarrow 0$$

Figure 5

Proof: The first statement will be a direct consequence of Theorem 6.11, below; the necessary assumption here that $\mathbb{F}_X$ has length $n-1$ is what makes $\mathbb{F}^{(X,X_U)}[-\mathbf{a}-\mathbf{1}](1-n)$ a *resolution* instead of just a free complex—that is, there are no terms in negative homological degrees. The construction of $\mathbb{F}^{(X,X_U)}$ from $\mathbb{F}_X$ preserves minimality because the matrices defining the differential of the former are submatrices of the transposes of those defining the latter, and we need only check that these entries are in $\mathfrak{m}$. $\square$

Remark 5.9 (i) The hypothesis of the theorem requires that X have dimension $(n-1)$, so that $\mathbb{F}_X$ has minimal *length*, but it does not require that $\mathbb{F}_X$ actually be a minimal resolution.

(ii) It can be shown that X_U may be replaced in the theorem by $X_U(\mathbf{b}+\mathbf{1})$ for any $\mathbf{b}$ satisfying $\mathbf{0} \preceq \mathbf{b} \preceq \mathbf{a}_I - \mathbf{a}_{I^\vee}$. Here, again, is the mysterious invariant from the remark after Corollary 2.14. In most cases of interest, though, $X_U = X_U(\mathbf{b})$ for all such $\mathbf{b}$.

Example 5.10 The free complex in Figure 5 is the minimal free resolution of the ideal $I^\vee$ from Example 1.8. The reader may check, for instance, that the product of the large matrix in Figure 5 with the list of generators for $I^\vee$ (which may be treated as a matrix with one row) is zero. Note that the homological and $\mathbb{Z}^n$-graded shifts promised by Theorem 5.8 aren't visible from the matrices. □

Theorem 5.8 affords a generalization of [3], Theorem 8.3 on reading irreducible decompositions off of cellular resolutions. We will need the following.

Lemma 5.11 *If the labelled cell complex X supports a minimal free resolution of an artinian ideal $J \subseteq S$ then X is pure of dimension $n-1$.*

Proof: Any facet has dimension > 0, so suppose that F is a facet of dimension $d > 0$. Denote by F^* the basis element of the cocellular complex $\mathbb{F}^X$. The modules $\underline{\mathrm{Ext}}^{\cdot}(J, S)$ can be calculated as the cohomology of $\mathbb{F}^X$ by definition, and the coboundary $\delta(F^*)$ is zero because F is a facet. Moreover, the image of δ is contained in $\mathfrak{m}\mathbb{F}^X$ by minimality of $\mathbb{F}_X$, whence F^* is not itself a coboundary. Thus F^* represents a nonzero element of $\underline{\mathrm{Ext}}^d(J, S) \cong \underline{\mathrm{Ext}}^{d+1}(S/J, S)$. It follows that $d = n-1$ because S/J has only one nonzero such $\underline{\mathrm{Ext}}$ module [5], Proposition 3.3.3(b)(i). □

For the statement of the next theorem, the following notation is convenient. Suppose $\mathbf{a} \succeq \mathbf{a}_I$ and define, for any $\mathbf{1} \preceq \mathbf{b} \preceq \mathbf{a}+\mathbf{1}$, the *bounded part* $\mathbf{b}_B := (\mathbf{a}+\mathbf{1}-\mathbf{b})^{\mathbf{a}}$ of $\mathbf{b}$ with respect to $\mathbf{a}$ to be the vector whose i^{th} coordinate is b_i if $b_i \le a_i$ and zero if $b_i = a_i + 1$.

Theorem 5.12 *Let $\mathbb{F}_X$ be a minimal cellular resolution of $I + \mathfrak{m}^{\mathbf{a}+\mathbf{1}}$. Then the facets of X are in bijection with the irredundant irreducible components of I, and the intersection $\bigcap_F \mathfrak{m}^{(\mathbf{a}_F)_B}$ over all facets $F \in X$ is an irredundant irreducible decomposition of I.*

Proof: It follows from Lemma 5.11 that under these conditions X must be pure of dimension $n-1$. Using this, it suffices to show that the label on any facet is $\succeq \mathbf{1}$, for then each facet corresponds to a minimal generator of $I^{\mathbf{a}}$ by Theorem 5.8 and we are done by Proposition 1.7. Suppose, then, that $\mathbf{a}_F$ is ≤ 0 in some coordinate for some facet F; say $(\mathbf{a}_F)_n = 0$. For $t \gg 0$ consider $Y := X_B(t, t, \ldots, t, 0)$, which gives a resolution of $J := (I + \mathfrak{m}^{\mathbf{a}+\mathbf{1}}) \cap k[x_1, \ldots, x_{n-1}]$ by [4], Corollary 1.3. The resolution $\mathbb{F}_Y$ is minimal because $\mathbb{F}_X$ is, and Y has dimension $n-1$ because $F \in Y$. On the other hand, J is an artinian ideal of $k[x_1, \ldots, x_{n-1}]$, which contradicts Lemma 5.11 (with n replaced by $n-1$). □

The major consequence of Theorem 5.8 is the construction of the cohull resolution (Definition 5.15) from the hull resolution [4], Section 2. Therefore, we recall here the definition of the hull complex. Let $t > (n+1)!$ and define $t^{\mathbf{b}} := (t^{b_1}, \ldots, t^{b_n})$. The convex hull of the points $\{t^{\mathbf{b}} \mid x^{\mathbf{b}} \in I\}$ is a polyhedron P_t whose face poset is independent of t. It is shown in [4] that the vertices of P_t are given by those $t^{\mathbf{b}}$ such that $x^{\mathbf{b}}$ is a minimal generator of I. The hull complex $\mathrm{hull}(I)$ is defined to be the bounded faces of P_t, but it may also be described as those faces of P_t admitting a strictly positive inner normal. The hull complex is labelled via the labels on its vertices.

Theorem 5.13 (Bayer-Sturmfels) *The free complex $\mathbb{F}_{\mathrm{hull}(I)}$ is a cellular resolution of I.* □

Example 5.14 Let Λ be the set of exponents on the minimal generators of a generic monomial ideal I, and let X be the labelled simplex with vertices in Λ. The *Scarf complex* of I is the labelled subcomplex $\Delta_I \subseteq X$ determined by

$$\Delta_I \ = \ \{F \in X \mid \mathbf{a}_F = \mathbf{a}_G \Rightarrow F = G\}.$$

It is minimal and coincides with the hull resolution of I by [4], Theorem 2.9. See Example 5.6. □

Definition 5.15 (The cohull resolution) *The* cohull resolution $\mathrm{cohull}_{\mathbf{a}}(I)$ *of an ideal I with respect to $\mathbf{a} \succeq \mathbf{a}_I$ is defined to be the relative cocellular resolution $\mathbb{F}^{(X,X_U)}[-\mathbf{a}-\mathbf{1}](1-n)$, where X is the hull complex of $I^{\mathbf{a}} + \mathfrak{m}^{\mathbf{a}+\mathbf{1}}$. The* canonical cohull resolution*, or simply* the cohull resolution cohull(I) *of I is obtained by taking $\mathbf{a} = \mathbf{a}_I$.*

The cohull resolution, like the hull resolution, is a possibly nonminimal resolution that preserves some of the symmetry (in the generators and irreducible components) of an ideal.

There are some geometric properties of hull resolutions of artinian ideals that make cohull resolutions a little more tangible. Suppose, for instance, that J is an artinian monomial ideal, with $x_1^{d_1}, \ldots, x_n^{d_n}$ among its minimal generators. Choose $t > (n+1)!$, and let $v_1, \ldots, v_n$ be the vertices of the polyhedron P_t determined by these minimal generators. The vertices $\{v_i\}$ of P_t span an affine hyperplane which will be denoted by H.

Fix a strictly positive inner normal φ_G for each $G \in \mathrm{hull}(J)$. Recall that P_t is contained in the (closed) polyhedron $\mathbf{1} + \mathbb{R}^n_+$ (since monomials in S have no negative exponents). Each face $G \in \mathrm{hull}(J)$ spans an affine space which does not contain the vector $\mathbf{1} \in \mathbb{R}^n$ because the hyperplane containing G and normal to φ_G does not contain $\mathbf{1}$. Therefore the projection π from the point $\mathbf{1}$ to the hyperplane H induces a homeomorphism $\mathrm{hull}(J) \to \pi(\mathrm{hull}(J))$. In fact,

Proposition 5.16 *If J is artinian, $\pi(\mathrm{hull}(J))$ is a regular polytopal subdivision of the simplex $H \cap P_t$.*

Proof: That $H \cap P_t \subset \mathbf{1} + \mathbb{R}^n_+$ is a simplex follows because it is convex and contains $v_1, \ldots, v_n$. Now π induces a map of the boundary $\partial P_t \to H \cap P_t$ which is obviously surjective. Suppose that $\pi(\mathbf{w})$ is in the interior of $H \cap P_t$ for some $\mathbf{w} \in \partial P_t$. It is enough to show that if a nonzero support functional φ attains its minimum on P_t at $\mathbf{w}$ then φ is strictly positive. All coordinates of φ are ≥ 0 *a priori* because it attains a minimum on P_t; but if the i^{th} coordinate of φ is zero then $\langle \varphi, v_i \rangle < \langle \varphi, \mathbf{w} \rangle$ and φ cannot be minimized at $\mathbf{w}$. □

Remark 5.17 *This generalizes the result [3], Corollary 4.5 for generic artinian monomial ideals, in view of [4], Theorem 2.9. Regular subdivisions here are as in [19], Definition 5.3.*

We arrive at the following characterization for artinian hull complexes:

Theorem 5.18 *If X is the hull complex of an artinian monomial ideal, then $|X|$ is a simplex and the negatively unbounded complex X_U is the topological boundary of X.*

Proof: By the previous proposition, it suffices to show that a face G of the hull complex of any (not necessarily artinian) ideal has a label without full support if and only if it is contained in the topological boundary of the shifted positive orthant $\mathbf{1} + \mathbb{R}^n_+$. But this holds because the i^{th} coordinate of $\mathbf{a}_G$ is zero if and only if every vertex of G (and hence every point in G) has i^{th} coordinate 1. □

Although cohull resolutions are relative cocellular by definition, they can frequently be viewed as cellular resolutions, as well. In fact, with a slight weakening of the notion of labelled cell complex, all cohull resolutions are weakly cellular. To be precise, define a *weakly labelled cell complex* to be the same as a labelled cell complex, except that instead of requiring that the label $\mathbf{a}_F$ be equal to the join $\bigvee_{v\in F} \mathbf{a}_v$, we require only that $\mathbf{a}_F \preceq \bigvee_{v\in F} \mathbf{a}_v$ whenever $\dim F > 0$. A free complex or resolution is called *weakly cellular* if it is supported on a weakly labelled cell complex.

Theorem 5.19 *The cohull resolution of I with respect to $\mathbf{a}$ is weakly cellular for any $\mathbf{a} \succeq \mathbf{a}_I$.*

Proof: Let $J = I + \mathfrak{m}^{\mathbf{a}+\mathbf{1}}$ and assume the notation from after Definition 5.15. Define Q_t to be the intersection of P_t with the closed half-space containing the origin and determined by the hyperplane H. Then Q_t is a polytope which may also be described as the convex hull of (all of) the vertices of P_t. Furthermore, the bounded faces of P_t are simply those faces of Q_t which admit a strictly positive inner normal. Thus $X := \mathrm{hull}(J)$ is a subcomplex of the boundary complex of Q_t, as is the boundary ∂X.

Let $Y \subset \partial Q_t$ be the subcomplex generated by the facets of Q_t whose inner normal is *not* strictly positive. Denote chain and relative cochain complexes over k by $\mathcal{C}_\cdot(-)$ and $\mathcal{C}^\cdot(-,-)$. Then $Y \cap X = \partial X$ and the $\mathcal{C}^\cdot(Q_t, Y) = \mathcal{C}^\cdot(X, \partial X)$. For elementary reasons, $\mathcal{C}^\cdot(Q_t, Y) \cong \mathcal{C}_\cdot(X^\vee)$ for some subcomplex $X^\vee$ of the polar polytope $Q_t^\vee$ (use, for instance, the methods of [19], Sections 2.2–2.3). Note that the isomorphisms will exist regardless of the incidence functions in question, by [5], Theorem 6.2.2. That $X^\vee$ is weakly labelled follows from the isomorphism $\mathcal{C}_\cdot(X^\vee) \cong \mathcal{C}^\cdot(X, \partial X)$ and the remark after Definition 5.3. Indeed, the condition $F \supseteq G \Rightarrow \mathbf{a}_F \succeq \mathbf{a}_G$ for faces $F, G \in (X, \partial X)$ is equivalent to the condition $F^\vee \subseteq G^\vee \Rightarrow -\mathbf{a}_F \preceq -\mathbf{a}_G$ for faces of $X^\vee$, and this need only be applied when F is a facet containing G and $F^\vee$ is a vertex of $G^\vee$. □

Proposition 5.20 *If a weakly cellular resolution is minimal, it is cellular. In particular, if a cohull resolution is minimal, it is cellular.*

Proof: Let $(\widetilde{\mathbb{F}}, \widetilde{\partial})$ denote the augmented complex $\mathbb{F}_X \to I \to 0$, where X is a weakly labelled complex supporting a free resolution of I. We show that if $G \in X$ then $\mathbf{a}_G \succ \bigvee_{v\in G} \mathbf{a}_v$ implies $\mathbb{F}$ is not minimal. This is vacuous if $\dim G = 0$, so assume $\dim G$ has minimal dimension ≥ 1, and suppose that $\mathbf{a}_G - \mathbf{e}_i \succeq \bigvee_{v \in G} \mathbf{a}_v$. Then $\widetilde{\partial}(G) = x_i y$ for some $y \in \widetilde{\mathbb{F}}$ because $\dim G$ is minimal. It follows that $x_i \widetilde{\partial}(y) = \widetilde{\partial}(x_i y) = 0$, whence $\widetilde{\partial}(y) = 0$ because $\widetilde{\mathbb{F}}$ is torsion-free. Thus $\widetilde{\partial}(G) \in x_i \ker(\widetilde{\partial}) \subseteq \mathfrak{m} \cdot \ker(\widetilde{\partial})$ does not represent a minimal generator of $\ker(\widetilde{\partial})$ by Nakayama's Lemma for graded modules. □

Remark 5.21 For cohull resolutions the proposition is probably true without the hypothesis of minimality, but a proof (which would likely be geometric instead of algebraic) has not been found. In particular, all cohull resolutions in the examples below are cellular. Cellularity of the cohull resolution is equivalent to the following more concrete statement: the label on any interior face of the hull complex of an artinian ideal is the greatest common divisor of the labels on the facets that contain it.

Example 5.22 *(continuation of Example 1.9)* The minimal resolution of the permutahedron ideal I of Example 1.9 is, by [4], Example 1.9, the hull resolution, which is supported on a permutahedron. The minimal resolution of $I + \mathfrak{m}^{(n+1)\mathbf{1}}$ is also the hull resolution, and is supported on the complex X which may described as follows.

There are two kinds of faces of X. The first kind are those that make up the boundary ∂X; these are indexed by the proper nonempty faces $F \in \Delta$ and have vertices $t^{(n+1)\mathbf{e}_i} \in P_t$ for $i \in F$ (recall from Section 1 that $\mathbf{e}_i$ denotes the i^{th} basis vector of $\mathbb{Z}^n$ and $\Delta = \{1, \ldots, n\}$ is the $(n-1)$-simplex). On the other hand, the interior p-faces of X are in bijection with the chains

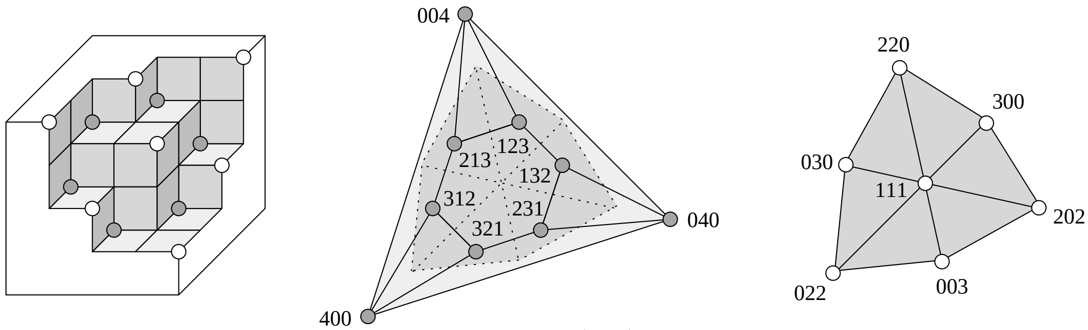
$$\emptyset \prec F_1 \prec F_2 \prec \cdots \prec F_{n-p} \tag{4}$$

of faces of Δ, where F_{n-p} might (or might not) equal Δ. Note that the interior faces of X for which $F_{n-p} = \Delta$ are faces of the permutahedron itself.

More generally, an interior p-face G given by (4) for which $F_{n-p} \neq \Delta$ is affinely spanned by the permutahedral $(p-1)$-face $G' : \emptyset \prec F_1 \prec \cdots \prec F_{n-p} \prec \Delta$ and the "artinian" vertices $\{t^{(n+1)\mathbf{e}_i} \mid i \notin F_{n-p}\}$ of P_t. In fact, a functional which attains its minimum (in P_t) on G may be produced directly. For this purpose, define for any $F \in \Delta$ the functional $F^\dagger$ on $\mathbb{R}^n$ to be the transpose of F; i.e. $\langle F^\dagger, \mathbf{e}_i \rangle = 1$ if $i \in F$ and zero otherwise. Then the functional $\varphi_\epsilon := \mathbf{1}^\dagger + \epsilon \sum_{j=1}^{n-p} F_j^\dagger$ attains its minimum (in P_t) on G' for all $0 < \epsilon \ll 1$. But for $\epsilon \gg 0$ we have $\langle \varphi_\epsilon, t^{(n+1)\mathbf{e}_i} \rangle < \langle \varphi_\epsilon, G' \rangle$ whenever $i \notin F_{n-p}$. Thus we can choose the unique ϵ that makes $\langle \varphi_\epsilon, t^{(n+1)\mathbf{e}_i} \rangle = \langle \varphi_\epsilon, G' \rangle$ for all $i \notin F_{n-p}$, so that φ_ϵ attains its minimum on G.

It is easy to check that the labels on the faces of X are distinct, whence $\mathbb{F}_X$ is the minimal resolution of $I + \mathfrak{m}^{(n+1)\mathbf{1}}$ by [4], Remark 1.4. In particular, the irredundant irreducible components of I are in bijection with the facets of X by Theorem 5.12, and the generators of the forest ideal $I^\vee$ are given by $x^{(n+1)\mathbf{1} - \mathbf{a}_G}$ for facets $G \in X$. This recovers the generators for $I^\vee$ in Example 1.9.

Retaining earlier notation, the face G has dimension $1 + \dim(G')$. Thus the p-faces of X are in bijection with the collection of p- and $(p-1)$-faces of the permutahedron. In fact, the (unlabelled)



I $\quad X = \mathrm{hull}(I + \mathfrak{m}^{(4,4,4)})$ $\quad X^\vee = \mathrm{cohull}(I^\vee)$

Figure 6: I and $I^\vee$ are the permutahedron and forest ideals when $n = 3$. The complex X is the (labelled) regular polytopal subdivision of the simplex promised by Proposition 5.16. Overlayed on this figure is the dual complex $X^\vee$ (without its labelling). At right, $X^\vee$ is shown with its labelling, which is $\mathbb{Z}^n$-shifted as per Theorem 5.8. Turn the picture over for the staircase of $I^\vee$.

pair $(|X|, |\partial X|)$ has the same faces as the pair $\Big(\partial(v * Y), v\Big)$ consisting of the boundary of the cone over the permutahedron Y rel the apex of the cone. The cellular complex $X^\vee$ supporting the cohull resolution of the forest ideal $I^\vee$ is therefore easy to describe. Let Y be the permutahedron in $\mathbb{R}^n$ and $Y^\vee$ its polar. Then $X^\vee$ is the cone over $\partial Y^\vee$ from the barycenter of $Y^\vee$. The vertices $G^\vee$ of $X^\vee$, which are labelled by the generators of $I^\vee$, almost all correspond to the facets G' of Y (whose labellings are as above). Only the apex of the cone is an exception, corresponding instead to the interior of Y. The case $n = 3$ is depicted in Figure 6; it should be noted that the equality $Y = Y^\vee$ is only because Y is 2-dimensional, not some more general self-duality.

Now $\mathrm{cohull}(I^\vee)$ is a cellular resolution of $I^\vee = I^\vee + \mathfrak{m}^{\mathbf{a}_I + \mathbf{1}}$, so we can dualize this cellular resolution using Theorem 5.8 again. This yields a minimal relative cocellular resolution of I, which is seen to be cellular and (coincidentally?) equal to $\mathrm{hull}(I)$. $\square$

Recall from Section 1 that an ideal is *cogeneric* if it is Alexander dual to a generic ideal. The minimal resolution of such an ideal was introduced in [15], where it was dubbed the *co-Scarf resolution*. The next theorem, along with the proof of Theorem 5.19 above, explains why the construction in [15] involved a subcomplex of the boundary of the simple polytope dual to the simplicial polytope of which the Scarf complex is a subcomplex. The theorem is a direct consequence of Theorem 5.8, Example 5.14, and Proposition 5.20.

Theorem 5.23 *Any cohull resolution of a cogeneric monomial ideal is minimal and cellular.* $\square$

Remark 5.24 That the co-Scarf resolution is cellular as opposed to weakly cellular was assumed in [4], Example 1.8 but overlooked in [15].

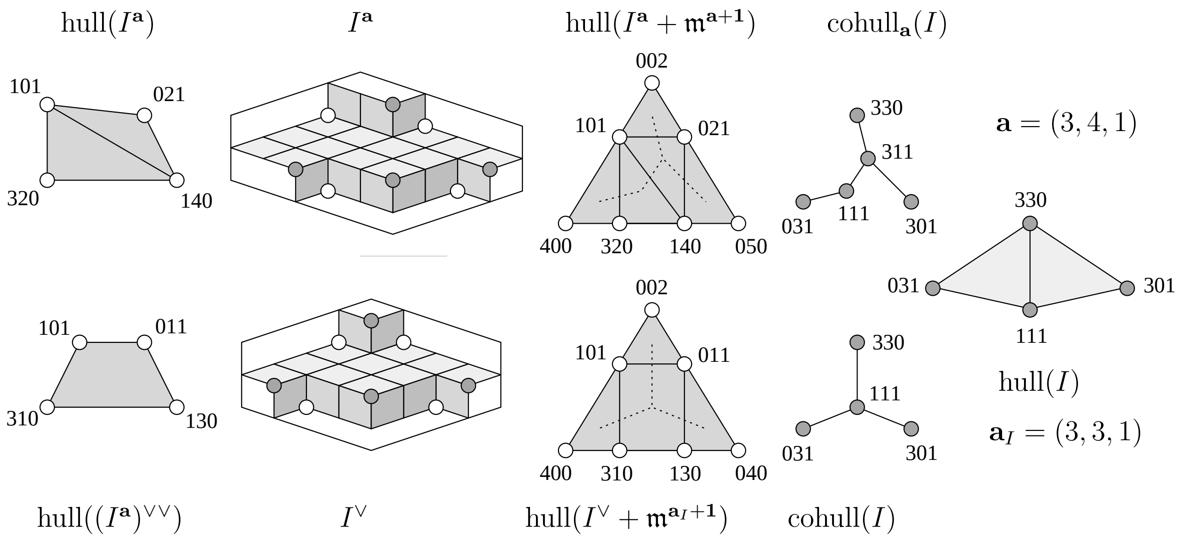


Figure 7

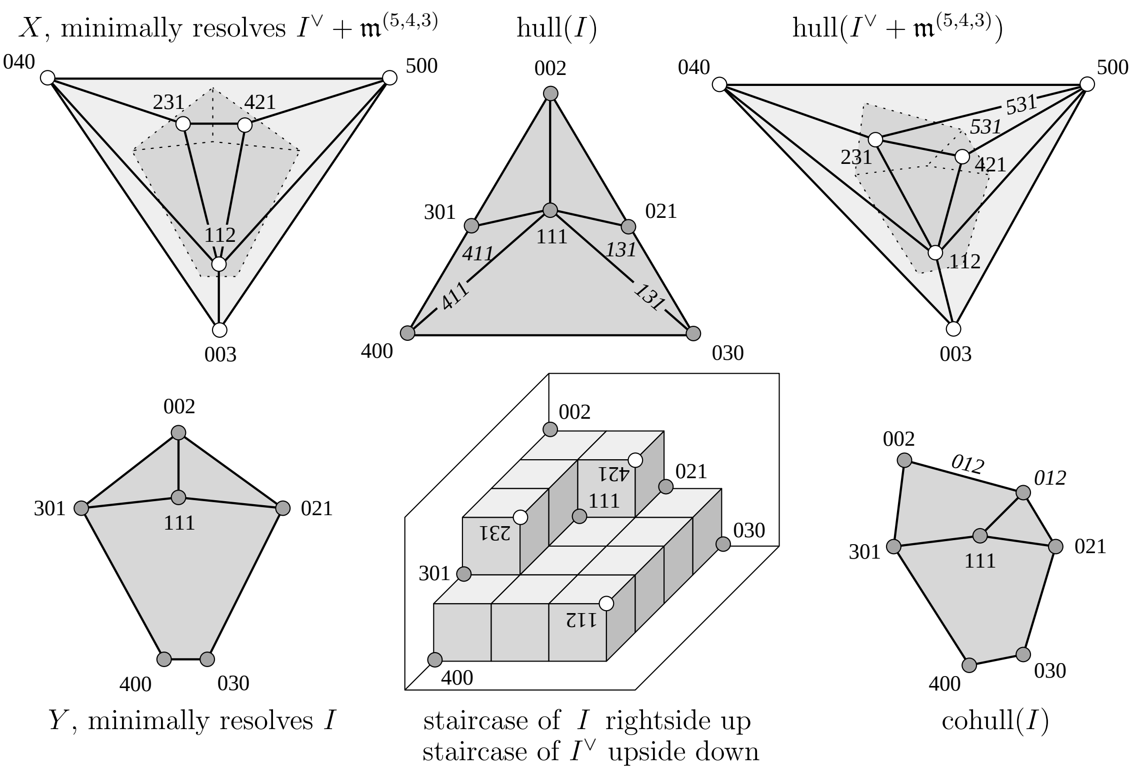


Figure 8

Example 5.25 It is possible for the hull and cohull resolutions to coincide for a given ideal I. For instance, this occurs if $I = \mathfrak{m}$; or if I is simultaneously generic and cogeneric (which turns out to be pretty hard to accomplish!); or if I is the permutahedron ideal in 3 variables. Conjecturally, the hull and cohull resolutions should coincide for permutahedron ideals of all dimensions. □

Example 5.26 Of course, it is also possible for the hull and cohull resolutions to be very different. For instance, the cohull resolution of the ideal $I^\vee$ from Examples 1.8 and 5.10 is the co-Scarf resolution, which is cellular and supported on an octagon with only one maximal face (dualize the picture in Figure 5). On the other hand, $\mathrm{hull}(I^\vee)$ is a triangulation of the same octagon. □

Example 5.27 The canonical cohull resolution can differ from a noncanonical cohull resolution. For instance, let $I = \langle x^3z, xyz, y^3z, x^3y^3\rangle$ and $\mathbf{a} = (3,4,1)$, so $I^{\mathbf{a}} = \langle xz, x^3y^2, xy^4, y^2z\rangle$ and $I^\vee = \langle xz, x^3y, xy^3, yz\rangle$. Since $\mathrm{hull}(I)$ is not minimal, we look elsewhere for the minimal resolution of I. But $\mathrm{hull}(I^{\mathbf{a}} + \mathfrak{m}^{\mathbf{a}+\mathbf{1}})$ is not minimal, and the failure of minimality occurs in such a way that $\mathrm{cohull}_{\mathbf{a}}(I)$ is also not minimal. On the other hand, the offending nonminimal edge is not present in $\mathrm{hull}(I^\vee + \mathfrak{m}^{\mathbf{a}_I+\mathbf{1}})$, and this resolution is minimal. It follows that $\mathrm{cohull}(I)$ is minimal. Note how the passage from $I^{\mathbf{a}}$ to $(I^{\mathbf{a}})^{\vee\vee} = I^\vee$ "tightens" the hull resolution of $I^{\mathbf{a}}$ to make the nonminimal edge disappear in $\mathrm{hull}((I^{\mathbf{a}})^{\vee\vee})$, cf. the remark after Corollary 2.14.

The labelled complexes supporting these resolutions are all depicted in Figure 7, where the resolutions with black vertices are drawn "upside down" to make their superimposition on the staircase diagram for I easier to visualize. Observe that a staircase diagram for I can be obtained by turning over the staircase diagram for either $I^{\mathbf{a}}$ or $I^\vee$, although these result in different "bounding boxes" for I. Note that all of the complexes, particularly the cohull complexes, are labelled and not just weakly labelled. □

Example 5.28 Finally, an example to illustrate that not all cellular resolutions come directly from hull and cohull resolutions, so that the algebraic techniques to prove exactness in Section 6 prove a stronger duality for resolutions than a geometric treatment such as that in [4] or [15] could provide. All of the labelled cellular complexes from this example are depicted in Figure 8.

Let $I = \langle z^2, x^3z, x^4, y^3, y^2z, xyz\rangle$, so that $I^\vee = \langle xyz^2, x^2y^3z, x^4y^2z\rangle$. Then $\mathrm{hull}(I)$ and $\mathrm{cohull}(I)$ are not minimal (the offending cells have italic labels); moreover, $\mathrm{cohull}_{\mathbf{a}}(I) = \mathrm{cohull}(I)$ for all $\mathbf{a} \succeq \mathbf{a}_I = (4,3,2)$. Nonetheless, the minimal resolution $\mathbb{F}_X$ of $I^\vee + \mathfrak{m}^{(5,4,3)}$ is cellular, so Theorem 5.8 applies, yielding a minimal relative cocellular resolution $\mathbb{F}^{(X,X_U)}[-(5,4,3)](-2)$ for I. In fact, this relative cocellular resolution is cellular, supported on the labelled cell complex Y. □

6 Deformations and limits of resolutions

The final item on the agenda is the proof of Theorem 5.8. To that end, the goal of this section is Theorem 6.11, which is actually a little more general than Theorem 5.8. It can be viewed as the result of applying a limiting process to a collection of pairs of linked artinian monomial ideals that are deformations of a given pair. The entire section is a setup to apply a limit to Proposition 3.11, and is another manifestation of the kinship of Alexander duality and other types of duality for Gorenstein rings. The maps $f_{\mathbf{b}}$ in the following definition accomplish the deformations.

Definition 6.1 *Define the map $f_{\mathbf{b}}: \mathbb{Z}^n \to \mathbb{Z}^n$ for $\mathbf{b} \succeq \mathbf{0}$ by the coordinatewise formula*

$$f_{\mathbf{b}}(\mathbf{c})_i = \begin{cases} c_i - b_i & \text{if } c_i \leq 0 \\ c_i & \text{if } c_i \geq 1 \end{cases}$$

To avoid messy exponents we also let $f_{\mathbf{b}}(x^{\mathbf{c}}) = x^{f_{\mathbf{b}}(\mathbf{c})}$. Whenever the symbol $f_{\mathbf{b}}$ is written, it will be assumed that $\mathbf{b} \succeq \mathbf{0}$.

Proposition 6.2 *If $I \subseteq S$ is any monomial ideal then $\langle f_{\mathbf{b}}(I)\rangle = S[\mathbf{b}] \cap \widetilde{I}$.*

Proof: It is clear from the definition that $f_{\mathbf{b}}(\mathbf{c}) \succeq -\mathbf{b}$ if $\mathbf{c} \succeq \mathbf{0}$, whence $\langle f_{\mathbf{b}}(I)\rangle \subseteq S[\mathbf{b}]$. Since also $f_{\mathbf{b}}(\mathbf{c})^+ = f_{\mathbf{b}}(\mathbf{c}^+)$, we conclude that $\langle f_{\mathbf{b}}(I)\rangle \subseteq \widetilde{I}$ as well. For the reverse inclusion, assume $x^{\mathbf{c}} \in S[\mathbf{b}] \cap \widetilde{I}$. Then $f_{\mathbf{b}}(x^{\mathbf{c}^+}) \in f_{\mathbf{b}}(I)$ and divides $x^{\mathbf{c}}$ because $f_{\mathbf{b}}(\mathbf{c}^+) \preceq \mathbf{c}$ whenever $\mathbf{c} \succeq -\mathbf{b}$, a fact which is easily seen from the definition. □

Recall from Section 5 that the join of $\mathbf{c}, \mathbf{c}' \in \mathbb{Z}^n$ is the componentwise maximum.

Lemma 6.3 *The map $f_{\mathbf{b}}$ preserves joins; that is, $f_{\mathbf{b}}(\mathbf{c} \vee \mathbf{c}') = f_{\mathbf{b}}(\mathbf{c}) \vee f_{\mathbf{b}}(\mathbf{c}')$.* □

This lemma is important because of the next proposition, originally due to D. Bayer. Let X be a labelled cell complex, and suppose $f: \mathbb{Z}^n \to \mathbb{Z}^n$ is a map respecting joins. Denote by $f(X)$ the labelled cell complex which is obtained by applying f to the labels on the faces of X. Thus $G \in f(X)$ is labelled by $f(\mathbf{a}_G)$ whenever $G \in X$ is labelled by $\mathbf{a}_G$.

Proposition 6.4 *Let $\mathbb{F}_X$ be a cellular resolution of a finitely generated module $M \subseteq T$. If $f: \mathbb{Z}^n \to \mathbb{Z}^n$ preserves joins then $\mathbb{F}_{f(X)}$ is a resolution of $\langle f(M)\rangle$.*

Proof: Note that because f respects joins the effect of f is determined by its effect on the vertex labels. Similarly, $\langle f(M)\rangle = \langle f(x^{\mathbf{b}}) \mid \mathbf{b}$ is a vertex label of $X\rangle$. Thus one only needs to check that $\mathbb{F}_{f(X)}$ is acyclic. It suffices to check that $X_B(\mathbf{b})$ is acyclic for all $\mathbf{b} \in \mathbb{Z}^n$, by the acyclicity criterion of [4], Proposition 1.2.

Suppose, then, that α is a cycle of the reduced chain complex of $|f(X)_B(\mathbf{b})|$. Then α also represents a cycle of $|X|$. Let $\mathbf{c}$ be the join of the labels on the faces in the support of α, considered as faces of X. Since f preserves joins, $f(\mathbf{c}) \preceq \mathbf{b}$ and $|X_B(\mathbf{c})| \subseteq |f(X)_B(\mathbf{b})|$. Now α is a boundary in the reduced chain complex of $|X_B(\mathbf{c})|$ by [4], Proposition 1.2, and it follows that α is also a boundary in the reduced chain complex of $|X_B(\mathbf{b})|$, completing the proof. □

Corollary 6.5 *If $\mathbb{F}_X$ is a cellular resolution of I then $\mathbb{F}_{f_{\mathbf{b}}(X)}$ is a cellular resolution of $S[\mathbf{b}] \cap \widetilde{I}$.* $\square$

Keeping the notation of the corollary we can augment $\mathbb{F}_{f_{\mathbf{b}}(X)}$ to a resolution of $S[\mathbf{b}]/S[\mathbf{b}] \cap \widetilde{I}$, homologically shifted down 1, by adding a summand $S[\mathbf{b}]$ in homological degree -1. We denote this augmented resolution by $\mathbb{F}_{\mathbf{b}}(X)$, and we let $\mathbb{F}^{\mathbf{b}}(X) := \mathbb{F}_{\mathbf{b}}(X)^*$, with differential $\delta_{\mathbf{b}}$. The generator of the summand $S[-f_{\mathbf{b}}(\mathbf{a}_F)] \subseteq \mathbb{F}_{\mathbf{b}}(X)$ corresponding to the face F will be denoted by $F_{\mathbf{b}}$, while the generator of $S[f_{\mathbf{b}}(\mathbf{a}_F)] = S[-f_{\mathbf{b}}(\mathbf{a}_F)]^* \subseteq \mathbb{F}^{\mathbf{b}}(X)$ will be denoted by $F^{\mathbf{b}}$. Keep in mind that $F^{\mathbf{b}}$ is in $\mathbb{Z}^n$-graded degree $-f_{\mathbf{b}}(\mathbf{a}_F)$.

We will soon be defining maps between the $\mathbb{F}^{\mathbf{b}}(X)$ for various $\mathbf{b}$, and the following lemma, particularly part (ii), will be the tool used to prove that these maps are well-defined, commute with the differentials, and form an inverse system.

Lemma 6.6 *If $\mathbf{b} \succeq \mathbf{b}' \succeq \mathbf{0}$ then*

$$\begin{array}{ll} (i) & f_{\mathbf{b}} = f_{\mathbf{b}-\mathbf{b}'} \circ f_{\mathbf{b}'}\,, \\ (ii) & f_{\mathbf{b}'}(\mathbf{c}) - f_{\mathbf{b}}(\mathbf{c}) = \mathbf{c} - f_{\mathbf{b}-\mathbf{b}'}(\mathbf{c})\,. \end{array}$$

Proof: Plug and chug, using the equality $f_{\mathbf{b}}(\mathbf{c})^+ = \mathbf{c}^+$ for (i). $\square$

Lemma 6.7 *For every $\mathbf{b} \succeq \mathbf{b}' \succeq \mathbf{0}$ we have an injection of chain complexes $\varphi_{\mathbf{b},\mathbf{b}'}\colon \mathbb{F}^{\mathbf{b}}(X) \hookrightarrow \mathbb{F}^{\mathbf{b}'}(X)$ sending $F^{\mathbf{b}}$ to $\dfrac{m_F}{f_{\mathbf{b}-\mathbf{b}'}(m_F)} F^{\mathbf{b}'}$.*

Proof: There are two aspects to the proof: (i) the given map is an injection of homologically graded modules which (as a map of $\mathbb{Z}^n$-graded modules) has degree $\mathbf{0}$, and (ii) the injections commute with the differentials. The first follows from the equality $-f_{\mathbf{b}}(\mathbf{a}_F) = -f_{\mathbf{b}'}(\mathbf{a}_F) + \mathbf{a}_F - f_{\mathbf{b}-\mathbf{b}'}(\mathbf{a}_F)$ which is easily seen to be equivalent to Lemma 6.6(ii) when $\mathbf{c} = \mathbf{a}_F$. The second is a longer calculation directly from the definition of the differentials $\delta_{\mathbf{b}}$ and $\delta_{\mathbf{b}'}$ of the chain complexes $\mathbb{F}_{\mathbf{b}}$ and $\mathbb{F}_{\mathbf{b}'}$.

The definitions imply that $\delta_{\mathbf{b}}$ is just the transpose of the differential from the cellular free complex as defined in [4]. Thus, $\delta_{\mathbf{b}}(F^{\mathbf{b}}) = \sum_{G\in X} \varepsilon(G,F)\frac{f_{\mathbf{b}}(m_G)}{f_{\mathbf{b}}(m_F)} G^{\mathbf{b}}$, where ε is the incidence function defining the differential of X. Note that $\varepsilon(G,F)$ is nonzero only if $G \supseteq F$. We have

$$\begin{aligned} \delta_{\mathbf{b}'} \circ \varphi_{\mathbf{b},\mathbf{b}'}(F^{\mathbf{b}}) &= \sum_{G\in X} \varepsilon(G,F)\frac{f_{\mathbf{b}'}(m_G)}{f_{\mathbf{b}'}(m_F)} \cdot \frac{m_F}{f_{\mathbf{b}-\mathbf{b}'}(m_F)}\, G^{\mathbf{b}'} \\ &= \sum_{G\in X} \varepsilon(G,F)\frac{m_G}{f_{\mathbf{b}-\mathbf{b}'}(m_G)} \cdot \frac{f_{\mathbf{b}}(m_G)}{f_{\mathbf{b}}(m_F)}\, G^{\mathbf{b}'} \\ &= \varphi_{\mathbf{b},\mathbf{b}'} \circ \delta_{\mathbf{b}}(F^{\mathbf{b}}), \end{aligned}$$

where the transition from the first line to the second is accomplished by two applications of Lemma 6.6(ii). $\square$

Lemma 6.8 *If* $\mathbf{b} \succeq \mathbf{b}' \succeq \mathbf{b}'' \succeq \mathbf{0}$ *then* $\varphi_{\mathbf{b},\mathbf{b}''} = \varphi_{\mathbf{b}',\mathbf{b}''} \circ \varphi_{\mathbf{b},\mathbf{b}'}$.

Proof: We need only check the equality as maps of modules. The proof again uses property (ii) from Lemma 6.6, and it involves manipulations similar to those in the proof of Lemma 6.7. □

These lemmata show that we have an inverse system of complexes of free modules, so it is natural now to take the inverse limit. With $\mathbb{F}^t(X) := \mathbb{F}^{t\cdot\mathbf{1}}(X)$ we can simplify a little since the inverse systems $\{\mathbb{F}^{\mathbf{b}}(X)\}_{\mathbf{b}\succeq 0}$ and $\{\mathbb{F}^t(X)\}_{t\in\mathbb{N}}$ are cofinal, so that their limits are the same. We take this opportunity to note that our inverse limits, when taken in the category of $\mathbb{Z}^n$-graded objects and degree zero maps, will be denoted by ${}^*\varprojlim$, and that S is complete in this category. Recall that, for our inverse system $\{\mathbb{F}^t(X)\}_{t\in\mathbb{N}}$ of chain complexes, for instance, this is defined as

$$ {}^*\varprojlim_t \mathbb{F}^t(X) \;=\; \bigoplus_{\mathbf{c}\in\mathbb{Z}^n} \varprojlim_t \mathbb{F}^t(X)_{\mathbf{c}}\,, $$

where the inverse limits on the right are in the category of chain complexes of k-vector spaces.

At each stage in the inverse system, $f_{\mathbf{b}}$ moves the labels on X_U away from the first orthant, in negative directions, turning any zeros into arbitrarily large negative integers (hence the name "negatively unbounded" for the subcomplex X_U of X). Then S-duality makes the negative integers positive. Thus the maps $f_{\mathbf{b}}$, combined as they are with S-duality in the definition of $\mathbb{F}^{\mathbf{b}}$, create irreducible components of $I^{\mathbf{a}}$ from those generators of I which do not have full support by pushing the zeros out to (positive) infinity. In the limit, the vertices defining those generators disappear. This provides the intuition for the next result.

Theorem 6.9 $\mathbb{F}^{(X,X_U)} = {}^*\varprojlim_t \mathbb{F}^t(X)$.

Proof: The first observations are that $\mathbb{F}^{(X,X_U)}$ is a subcomplex of $\mathbb{F}^t$ for all t, and that the maps $\varphi_{t,t'} := \varphi_{t\cdot\mathbf{1},t'\cdot\mathbf{1}}$ defining the inverse system restrict to the identity on $\mathbb{F}^{(X,X_U)}$. This is because of the way $f_t := f_{t\cdot\mathbf{1}}$ is defined:

$$ f_{t-t'}(m_F) = m_F \iff t = t' \text{ or } F \notin X_U \tag{5} $$

because $f_{\mathbf{b}}(\mathbf{c})_i = c_i$ for all i precisely when $\mathbf{c} \succeq \mathbf{1}$. Thus we have, for all $t \geq 0$, exact sequences

$$ 0 \to \mathbb{F}^{(X,X_U)} \to \mathbb{F}^t(X) \to \mathbb{F}^t(X_U) \to 0 \tag{6} $$

giving rise to a corresponding exact sequence of inverse systems. To be more precise, the maps $\{\varphi_{t,t'}\}$ from the inverse system $\{\mathbb{F}^t(X)\}_{t\in\mathbb{N}}$ induce maps $\{\psi_{t,t'}\colon \mathbb{F}^t(X_U) \to \mathbb{F}^{t'}(X_U)\}_{t\geq t'}$ which make $\{\mathbb{F}^t(X_U)\}_{t\in\mathbb{N}}$ into an inverse system.

It is readily seen that the maps $\psi_{t,t'}$ are injections, so that ${}^*\varprojlim_t \mathbb{F}^t(X_U) = \bigcap_t \psi_{t,0}\Big(\mathbb{F}^t(X_U)\Big)$. Furthermore, statement (5) implies that $\psi_{t,t'}\Big(\mathbb{F}^t(X_U)\Big) \subseteq \mathfrak{m}\mathbb{F}^{t'}(X_U)$ if $t > t'$. It follows from the

Krull intersection theorem that the inverse limit is zero. Since the inverse limit is always left exact our exact sequence of inverse systems arising from (6) yields the desired isomorphism. □

So we can write $\mathbb{F}^{(X,X_U)}$ as an inverse limit. What have we gained? In the category of $\mathbb{Z}^n$-graded objects in which each graded piece has finite dimension over k (e.g. if the objects are chain complexes which are finitely generated as S-modules), the functor ${}^*\varprojlim$ is exact, at least in the case where the inverse systems are indexed by $\mathbb{N}$—see [18], Exercise 3.5.2. With this in mind the following corollary is a simple consequence of [18], Theorem 3.5.8.

Corollary 6.10 *To compute homology we have $H_i(\mathbb{F}^{(X,X_U)}) = {}^*\varprojlim_t H_i(\mathbb{F}^t(X))$.* □

Until this point in this section, the labelled cell complex X has been arbitrary. Now, however, we suppose that X supports a cellular free resolution of the ideal $I + \mathfrak{m}^{\mathbf{a}+\mathbf{1}}$, with $\mathbf{a} \succeq \mathbf{a}_I$. We will see shortly that for any t the only nonvanishing homology of $\mathbb{F}^t(X)$ is in homological degree $1-n$, so the previous corollary implies that the same holds for $\mathbb{F}^{(X,X_U)}$. Now $\mathbb{F}_X$ has length at least $n-1$ (i.e. $\dim X \geq n-1$) because it gives a free resolution of an artinian ideal; if we are so lucky that $\mathbb{F}_X$ has length exactly $n-1$, then the summand of $\mathbb{F}^{(X,X_U)}$ in homological degree $1-n$ will be the last nonzero term. In other words, $\mathbb{F}^{(X,X_U)}$ will be a *free resolution* of some S-module. This is what makes Theorem 5.8 a special case of the next result. Even if we aren't so lucky with the length of $\mathbb{F}_X$, at least it will be split exact in homological degrees $> n-1$ (so that $\mathbb{F}^{(X,X_U)}$ is split exact in homological degrees $< 1-n$), and we can still determine what the nonzero homology module is:

Theorem 6.11 *Under the above conditions, $H_i(\mathbb{F}^{(X,X_U)}) = 0$ if $i \neq 1-n$, and $H_{1-n}(\mathbb{F}^{(X,X_U)}) = I^{[\mathbf{a}]}[\mathbf{a}+\mathbf{1}]$.*

Proof: Let $J = I + \mathfrak{m}^{\mathbf{a}+\mathbf{1}}$. For any $\mathbf{b} \succeq \mathbf{0}$ Corollary 6.5 implies that $\mathbb{F}_{\mathbf{b}}(X)$ is a free resolution of the module $S[\mathbf{b}]/\,S[\mathbf{b}]\cap\widetilde{J}$, homologically shifted down by 1. Thus $\mathbb{F}^{\mathbf{b}}(X)$, which is the S-dual of $\mathbb{F}_{\mathbf{b}}(X)$, is a complex whose homology in degree $i-1$ is $\underline{\mathrm{Ext}}^i_S\Big(S[\mathbf{b}]/\,S[\mathbf{b}]\cap\widetilde{J},\, S\Big)$. Now $S[\mathbf{b}]/\,S[\mathbf{b}]\cap\widetilde{J} \subseteq T/\widetilde{J}$ is artinian since $J = I + \mathfrak{m}^{\mathbf{a}+\mathbf{1}}$ is, and it is noetherian because $S[\mathbf{b}]$ is. Hence the Ext module in question is, by [5], Theorem 3.3.10(c), nonzero only for $i = n$. Moreover, Proposition 3.11 produces the equality

$$\underline{\mathrm{Ext}}^n_S\Big(S[\mathbf{b}]/\,S[\mathbf{b}]\cap\widetilde{J}, S\Big) \;=\; \Big(I\,/\,I\cap\mathfrak{m}^{\mathbf{a}+\mathbf{b}+\mathbf{1}}\Big)[\mathbf{a}+\mathbf{1}].$$

Taking the ${}^*\varprojlim_{\mathbf{b}}$ of this last line and applying Corollary 6.10 along with the completeness of S proves the theorem. □

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