ARTICLE

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An open letter concerning

Explanation of low Hurst exponent for Riemann zeta zeros

O. Shanker

In 2006 a striking result was published (Generalised Zeta Functions and Self-Similarity of Zero Distributions, J. Phys. A 39(2006), 13983-13997) about the statistics of the zeros of the Riemann zeta function. The paper (by the current author) applied rescaled range analysis, and found that the zeros exhibited an unusably low Hurst Exponent. While that paper discussed some possible explanations, no clear reason for the low Hurst Exponent emerged. This was particularly interesting because the most obvious explanation, that the differences of the zeros have a very large anti-correlation, would be very unusual indeed.

Recently the author came up with empirical evidence for another less radical explanation, and submitted the explanation for publication, and it was rejected. The main reason for the rejection appears to be the limited content. While the content is empirical and limited to a single point, the author nevertheless feels that the result should be published, not as an important new result, but because it provides a plausible explanation for the original findings. The referee also commented that the author did not mention that the distribution of the fluctuating part of the zero counting function has been known to be Gaussian distributed, and did not explain what the Hurst exponent was. The author agrees that the paper may benefit by having more detail and references. However, the original article had detailed references and discussion about the literature concerning the distribution of the zeros of the Riemann zeta function, so the reader may rely on the original paper for details and references. The author is not aware of any errors in the paper, and has not made any changes in the paper.

The referee also mentioned that there may be many distributions other than the Gaussian that would produce the same results. The author agrees: the key point of the paper is that the rescaled range analysis does not necessarily imply a long-range anti-correlation in the differences of the zeros, and to give an indication of other possible, less radical explanations.

Explanation of low Hurst exponent for Riemann zeta zeros

O. Shanker *

Abstract

We discuss a possible explanation for the low Hurst exponent extracted from a rescaled range analysis of the large height Riemann zeta function zeros.

Physicists have studied the zeros of the Riemann zeta function because of its relation to the spectra of random matrix theories (RMT) [1, 2, 3, 4] and the spectra of classically chaotic quantum systems [5, 6, 7, 8]. A rescaled range analysis of the large height Riemann zeta zeros leads to a very low Hurst exponent (~ 0.1) over several orders of magnitude variation (from 10^7 to 10^{22}) in the heights of the zeros [9]. So far there has been no explanation for this behaviour. One possible explanation is that the zeros represent a long range anti-correlation. However, one has to be careful in coming to that conclusion, since such a large anti-correlation is rather special [10]. In this work we argue that the low Hurst exponent is not due to an anti-correlation, but can be explained by superposing a random Gaussian correction term to Riemann's approximation for the variation of the number of zeros with height.

We first briefly set up the notation. The Riemann Zeta function is defined for $\mathrm{Re}(s) > 1$ by

$$\zeta(s) \,=\, \sum_{n=1}^{\infty} n^{-s} \,=\, \prod_{p} \,\left(1 - p^{-s}\right)^{-1}. \tag{1}$$

$\zeta(s)$ has a continuation to the complex plane and satisfies a functional equation

$$\xi(s) := \pi^{-s/2}\,\Gamma(s/2)\,\zeta(s) \,=\, \xi(1-s); \tag{2}$$

$\xi(s)$ is entire except for simple poles at $s = 0$ and 1. We write the zeroes of $\xi(s)$ as $1/2 + i\gamma$. The Riemann Hypothesis [11, 12, 13, 14] asserts that γ is real for the non-trivial zeroes. We order the γs in increasing order, with

$$\ldots\ldots\gamma_{-1} \,<\, 0 \,<\, \gamma_1 \,\leq\, \gamma_2 \ldots. \tag{3}$$

Then $\gamma_j \,=\, -\gamma_{-j}$ for $j = 1, 2, \ldots$, and γ_1, γ_2, $\ldots$ are roughly 14.1347, 21.0220, $\ldots$. The Hurst exponent is extracted by applying rescaled range analysis to the distribution of the spacings [9] $\delta_j \,=\, \gamma_{j+1} \,-\, \gamma_j$.

From Riemann's time it is known that the mean number of zeros with height less than γ (the smoothed Riemann zeta staircase) is approximately [14, 6]

$$< \mathcal{N}_{\mathcal{R}}(\gamma) >= (\gamma/2\pi)(ln(\gamma/2\pi) - 1) + \frac{7}{8}. \tag{4}$$

Thus, the mean spacing of the zeros at height γ is $2\pi(\ln(\gamma/2\pi))^{-1}$. Eqn. 4 can be inverted to give an approximation $\gamma_{a,i}$ for the i^{th} zero of the Riemann zeta function, if we set $< \mathcal{N}_{\mathcal{R}} >= i$. Let us write the i^{th} zero γ_i as

$$\gamma_i = \gamma_{a,i} + X_i, \tag{5}$$

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Table 1: Hurst Exponent for Riemann zeroes and for the values from Eqn. 5 with two different seed values for the random number generator. The sample size is 10000 and the standard deviation for X_i in Eqn. 5 is 0.25

Range of zeroes	Riemann zeros	Eqn. 5 seed A	Eqn. 5 seed B
$10^4 < j \leq 2 * 10^4$	0.51	0.51	0.51
$10^5 < j \leq 10^5 + 10^4$	0.21	0.21	0.20
$10^6 < j \leq 10^6 + 10^4$	0.10	0.13	0.11
$10^{12} < j \leq 10^{12} + 10^4$	0.12	0.12	0.11
$10^{22} < j \leq 10^{22} + 10^4$	0.12	0.12	0.11

where X_i is a correction term. We show that the observed behaviour of the rescaled range analysis is reproduced even if there is no anti-correlation between the Riemann zeta zeros, all that is needed is to assume that the correction term X_i is distributed normally. With this assumption, Table 1 shows the Hurst exponent extracted from the Riemann zeta zeros, and for comparision the Hurst exponent for the sequence given by Eqn. 5 with X_i distributed normally. We ran the analysis using two different seed values for the random number generator. The standard deviation of the normal distribution was taken to be 0.25 . We see from the table that Eqn. 5 reproduces the observed behaviour of the Riemann zeros fairly well. It is also not too sensitive to the assumed seed value used to generate the Gaussian term X_i in Eqn. 5. Finally, it reproduces the observed low Hurst exponent for the large height zeros, since for these zeros the first term in Eqn. 5 becomes essentially linear, and the Hurst exponent is then determined by the normal term, independent of the height. Thus, it appears that the observed low value for the large height Riemann zeta zeros is not due to an anti-correlation, but instead it is due to the fact that for these zeros the dependence on height is essentially linear, with a normal correction term superposed on the linear variation.

Figure 1 shows the rescaled range analysis for the zeros in the range $100000 \ldots 110000$. The horizontal axis is the log of the bin size used for the rescaled range analysis, and the vertical axis is the log of the mean rescaled range for the given values of bin size. The low slope at the low values of the bin size is due to the X_i term in Eqn. 5, and the increase in slope at higher bin sizes is due to the $\gamma_{a,i}$ term in Eqn. 5. Eqn. 5 gives a fairly good representation of the rescaled range behaviour of the actual Riemann zeta zeros.

In conclusion, we have presented evidence that the remarkable behaviour of the Riemann zeta zeros under rescaled range analysis can be explained by Riemann's approximation for the variation of the number of zeros with height coupled with a random Gaussian correction term.

References

[1] E. Wigner, "Random Matrices in Physics," *Siam Review*, **9**, 1-23, (1967).

[2] M. Gaudin, M. Mehta, "On the Density of Eigenvalues of a Random Matrix," *Nucl. Phys.*, **18**, 420-427, (1960).

[3] M. Gaudin, "Sur la loi Limite de L'espacement de Valuers Propres D'une Matrics Aleatiore," *Nucl. Phys.*, **25**, 447-458, (1961).

[4] F. Dyson, "Statistical Theory of Energy Levels III," *J. Math. Phys.*, **3**, 166-175, (1962).

[5] M. V. Berry, "Semiclassical theory of spectral rigidity," *Proc. R. Soc.*, **A 400** , 229-251, (1985).

[6] M. V. Berry, "Riemann's zeta function: a model for quantum chaos?," *Quantum chaos and statistical nuclear physics (Springer Lecture Notes in Physics)*, **263** , 1-17, (1986).

[7] M. V. Berry, "Quantum Chaology," *Proc. R. Soc.* , **A 413** , 183-198, (1987).

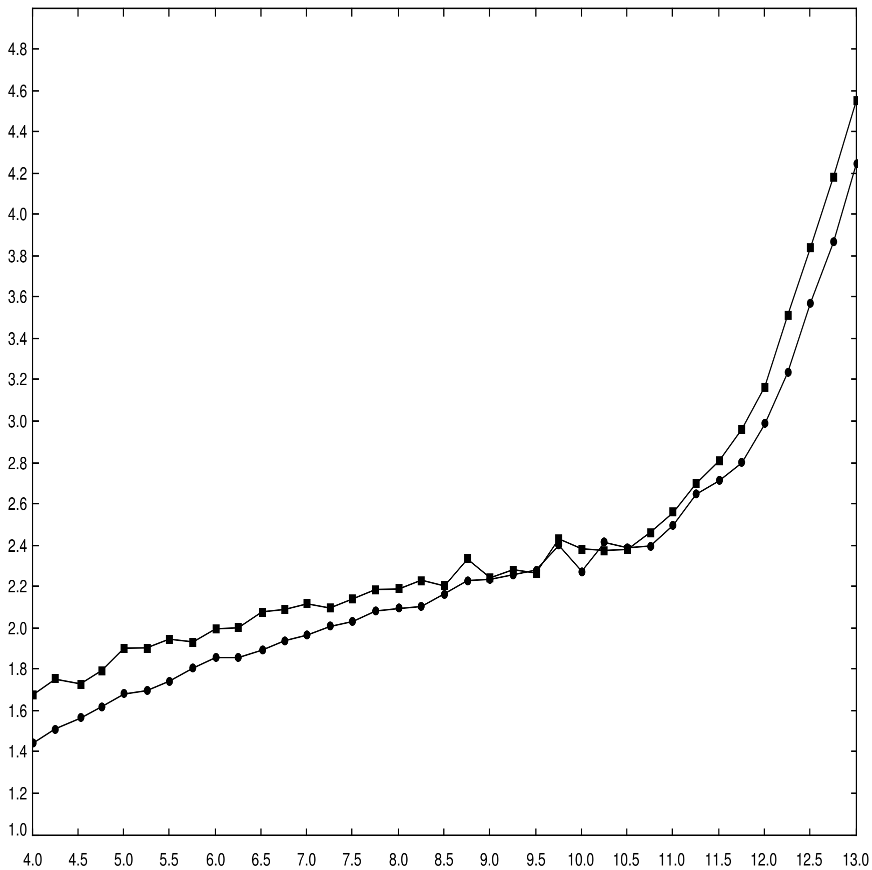

Figure 1: Rescaled range analysis for the zeros in the range 100000 . . . 110000. The y axis is the log of the rescaled range and the x axis is the log of the bin size. Squares represent the values for the Riemann zeros and circles represent the values for the sequence in Eqn. 5

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[8] M. V. Berry, ‘Number variance of the Riemann zeros‘,” *NonLinearity* , **1** ,399-407 , (1988).

[9] O. Shanker, Generalised Zeta Functions and Self-Similarity of Zero Distributions, *J. Phys. A* **39**(2006), 13983-13997.

[10] Govind Rangarajan and Mingzhou Ding, “An Integrated Approach to the Assessment of Long Range Correlation in Time Series Data”, *Phys. Rev.*, **E61**, 4991-5001, (2000).

[11] B. Riemann, “Über die Anzahl der Primzahlen uter
Einer Gegebenen Gröbe,” *Montasb. der Berliner Akad.*, (1858), 671-680.

[12] B. Riemann, “Gesammelte Werke”, Teubner, Leipzig, (1892).

[13] E. Titchmarsh, “The Theory of the Riemann Zeta Function,” Oxford University Press, Second Edition, (1986).

[14] H. M. Edwards, “Riemann’s Zeta Function,” Academic Press, (1974).