ARTICLE

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Extended real number system in measure theory
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# An open letter concerning

## *Extended real number system in measure theory*

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This article was originally submitted under the title "Extended real number system". Among the reasons given for its rejection was that there was undue focus on methods involved, such as consideration of separate cases, and that too little had been said about the relationship between what the review called "the proposed system" and other number systems that include infinity.

However, *no new system is being proposed* in the article and the very first sentence of the second paragraph includes the phrase "is usually defined as" in order to prevent any misinterpretation in this regard. But evidently to no avail.

Referring to the rule that infinity times zero should be zero as the rule "you propose," the review agreed with my observation that this does not work out well in many situations. However, the rule in question is quite standard in the extended real number system used for measure and integration, and nowhere does the article suggest that all conceivable systems are being studied under one roof.

The focus on multiplicity of cases has been argued for in the body of the article: Since multiplication in extended reals is defined by separating positive reals, negative reals, zero, positive infinity and negative infinity, verifying associativity alone requires an enormous number of dissimilar cases to be considered. The author feels that a construction procedure for the extended reals, involving only a manageable number of dissimilar cases—and this is what the article is mainly about, though not exclusively—is worth having on record.

Unfortunately, this is the only issue that is emphasized in the abstract. Within the article however, it has been pointed out that there is legitimate cause to question the consistency of a system having the usual properties which are assumed to hold for extended reals, and furthermore, that doubts arising on this score have been laid at rest in the rest of the discussion. The author feels that this is another feature that makes the exposition worth placing on record.

Besides a change in the title so as to include the phrase "in Measure Theory," there is an amendment in the abstract that reads "For the extended reals as used in measure theory (product $0 \cdot \infty$ is 0)" in place of "As an alternative." Also, the word "dissimilar" has been inserted.

# Extended real number system in measure theory

Satish Shirali*

**Abstract**

The extended real number system is usually defined by appending two new elements and stating rules of addition and multiplication for them. The associative and distributive laws are then supposed to be verified case by case; however, the number of cases to be verified is well over sixty. For the extended reals as used in measure theory (product $0 \cdot \infty$ is 0), we offer a construction through equivalence classes, in which the number of dissimilar cases does not exceed five at any stage.

In any proof that requires consideration of separate cases, usually the number of cases is small and it is quite clear that all of them have been taken into account. However, when the number of cases is large, it may not be so clear that none have been left out. For example, verifying the associative law for a binary operation described in tabular form on a set of $n$ elements is impractical when $n$ exceeds 4. Another such instance is that of the associative and distributive laws in the extended real number system.

The extended real number system $\tilde{\mathbb{R}}$ is usually defined as the union of $\mathbb{R}$ with two elements, written $\infty$ and $-\infty$, and endowed with the structure described in (a)—(h) below, in addition to that already available on its subset $\mathbb{R}$:

(a) $-\infty < x < \infty$ for every $x \in \mathbb{R}$;
(b) $x + \infty = \infty + x = \infty$ and $x + (-\infty) = (-\infty) + x = -\infty$ for every $x \in \mathbb{R}$;
(c) $\infty + \infty = \infty$ and $(-\infty) + (-\infty) = -\infty$;
(d) $\infty \cdot \infty = (-\infty) \cdot (-\infty) = \infty$ and $(-\infty) \cdot \infty = \infty \cdot (-\infty) = -\infty$;
(e) $-(-\infty) = \infty$ and $-(\infty) = -\infty$;
(f) $x \cdot \infty = \infty \cdot x = \infty$ and $x \cdot (-\infty) = (-\infty) \cdot x = -\infty$
for every positive $x \in \mathbb{R}$;
(g) $x \cdot (\infty) = (\infty) \cdot x = -\infty$ and $x \cdot (-\infty) = (-\infty) \cdot x = \infty$
for every negative $x \in \mathbb{R}$;
(h) $(\pm\infty) \cdot 0 = 0 \cdot (\pm\infty) = 0$.

In contexts other than measure and integration, one may wish to omit (h) and take the products occurring in it as "undefined". For instance, it is omitted in [1, p.12] with the consequence on p.314 that the Lebesgue integral of the identically zero function on $\mathbb{R}$ is left undefined by (53), considering that (49) requires the function to be written as zero times the characteristic function of $\mathbb{R}$. However, the same author includes (h) in [2, p.19].

Without (e), there would seem to be no basis for the common practice of regarding $x - (-\infty)$ as meaning $x + \infty$ and $x - (\infty)$ as meaning $x + (-\infty)$. We have therefore chosen to state it explicitly although most authors prefer not to.

It is immediate from the properties (a)—(h) that addition and multiplication in $\tilde{\mathbb{R}}$ are commutative. However, the associative law of addition and the distributive law, which continue to be valid under the restriction that $\infty$ and $-\infty$ do not both appear in any of the sums involved, are supposed to be verified case by case. The associativity of multiplication can be verified, again case by case, to be valid without restriction.

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The multiplication described by (a)—(h) is, in effect, a binary operation on a set of five elements, namely, positive real, negative real, 0, $\infty$ and $-\infty$ . In checking associativity therefore, one would have to consider 125 triplets $(x, y, z)$; however, 27 of these involve only real numbers and need not be checked. The remaining 98 can be reduced to 68 by taking advantage of the obvious commutativity, but this is still an uncomfortably large number of cases to handle.

Consequently, in any effort at a case-by-case verification of the associative and distributive laws in the extended real number system, it would be a legitimate concern whether all cases have actually been taken into account or not.

We have avoided adding the requirement that $\frac{x}{\infty} = 0$ for $x \in \mathbb{R}$ so as to keep clear of the consequence that

$$0 = 0{\cdot}\infty = \tfrac{1}{\infty} \cdot \infty \text{ but } \tfrac{1\cdot\infty}{\infty} \text{ is undefined.}$$

However, this observation raises a second concern, namely, whether (a)—(h) already contain a contradiction, even without this requirement.

With a view to addressing both concerns, we outline a method of "constructing" $\tilde{\mathbb{R}}$ from $\mathbb{R}$, in which we describe $\infty$ and $-\infty$ set theoretically in terms of $\mathbb{R}$ rather than pull them out of the sky, and moreover, the associative and distributive laws become transparent with just two cases each. The definitions of addition and multiplication in $\tilde{\mathbb{R}}$ undoubtedly call for separate cases to be considered, but it is transparent that none are left out.

We begin by describing what the objects $\infty$ and $-\infty$ are.

Let $\infty$ denote the class of real sequences "diverging to $\infty$" in the usual sense (no circularity involved in this) and $-\infty$ denote the obvious analogous class. Furthermore, for each $x \in \mathbb{R}$, let $[\![x]\!]$ denote the class consisting of a single sequence, namely, the constant sequence with each term equal to $x$. Set $[\![\mathbb{R}]\!] = \{[\![x]\!] : x \in \mathbb{R}\}$ and $\tilde{\mathbb{R}} = [\![\mathbb{R}]\!] \cup \{-\infty, \infty\}$. Then each element of $\tilde{\mathbb{R}}$ is a class of sequences and the classes are disjoint.

For $\alpha,\beta \in \tilde{\mathbb{R}}$, define $\alpha < \beta$ to mean: for any sequences $\{a_n\} \in \alpha$ and $\{b_n\} \in \beta$, the inequality $a_n < b_n$ holds for all sufficiently large $n$. Then it is easy verify that

$$[\![x]\!] < [\![y]\!] \Leftrightarrow x < y \;\text{ if } x, y \in \mathbb{R} \tag{1}$$

Also, $-\infty < \alpha < \infty$ for all $\alpha \in [\![\mathbb{R}]\!]$, and $-\infty < \infty$. Thus (a) holds with $x$ replaced by $[\![x]\!]$.

Suppose $\alpha,\beta \in \tilde{\mathbb{R}}$, and $\alpha \neq -\infty \neq \beta$. It is easy to see that the following four cases are exhaustive:

$$\text{(i)}\alpha, \beta \in [\![\mathbb{R}]\!] \quad \text{(ii)}\alpha \in [\![\mathbb{R}]\!] \text{ and } \beta = \infty \quad \text{(iii)}\alpha = \infty \text{ and } \beta \in [\![\mathbb{R}]\!] \quad \text{(iv)}\alpha, \beta = \infty.$$

It is equally straightforward to see in each of the four cases that, for any sequences $\{a_n\} \in \alpha$ and $\{b_n\} \in \beta$, the related sequence $\{a_n + b_n\}$ belongs to a unique $\gamma \in \tilde{\mathbb{R}}$. Therefore we may define $\alpha + \beta$ to be this unique $\gamma \in \tilde{\mathbb{R}}$.

In the course of arguing for the unique $\gamma$, it is also seen that

$$[\![x]\!] + \infty = \infty + [\![x]\!] = \infty \;\text{ if }\; x \in \mathbb{R},$$

$$\infty + \infty = \infty$$

and

$$[\![x]\!] + [\![y]\!] = [\![x + y]\!] \;\text{ if } x, y \in \mathbb{R} \tag{2}$$

One can proceed analogously when $\alpha,\beta \in \tilde{\mathbb{R}}$, and $\alpha \neq \infty \neq \beta$. All this establishes that $\alpha + \beta$ is uniquely defined except when one of them is $\infty$ and the other is $-\infty$ and that addition satisfies (c) as well as the properties claimed for it in (b), but with $x$ replaced by $[\![x]\!]$.

Now consider $\alpha,\beta \in \tilde{\mathbb{R}}$. When $\alpha,\beta \in [\![\mathbb{R}]\!]$, any sequences $\{a_n\} \in \alpha$ and $\{b_n\} \in \beta$ must be constant sequences and it is immediate that the related (constant) sequence $\{a_n b_n\}$ belongs to a unique $\delta \in [\![\mathbb{R}]\!] \subseteq \tilde{\mathbb{R}}$. When $\alpha = \infty$, the following five cases for $\beta$ are exhaustive:

$$[\![0]\!] < \beta \in [\![\mathbb{R}]\!]\,, \quad [\![0]\!] > \beta \in [\![\mathbb{R}]\!]\,, \quad [\![0]\!] = \beta, \quad \beta = \infty, \quad \beta = -\infty.$$

In each of the five cases, it is easy to arrive at the conclusion that: for any sequences $\{a_n\} \in \alpha$ and $\{b_n\} \in \beta$, the related sequence $\{a_n b_n\}$ belongs to a unique $\delta \in \tilde{\mathbb{R}}$. We note that it is essential here for $\beta = [\![0]\!]$ to consist of only the constant sequence $\{0,0,\dots\}$. Similarly when $\alpha = -\infty$. In view of the commutativity of multiplication in $\mathbb{R}$, the same conclusion can be drawn for the ten cases when $\beta = \pm\infty$. (It is inessential to the argument that the actual number of distinct cases is not 20 but 16, because the four cases when $\alpha = \pm\infty$ and $\beta = \pm\infty$ will occur twice among the 20.) Thus all cases have been covered and the aforementioned conclusion holds for all $\alpha,\beta \in \tilde{\mathbb{R}}$. Therefore we may define $\alpha\beta$ to be this unique $\delta \in \tilde{\mathbb{R}}$.

In the course of arguing for the unique $\delta$, it is also seen that multiplication satisfies

$$[\![x]\!]\,[\![y]\!] = [\![xy]\!] \text{ if } x, y \in \mathbb{R} \tag{3}$$

as well as (d),(h), and that it further satisfies (f),(g), with $x$ replaced by $[\![x]\!]$.

For any $\alpha \in \tilde{\mathbb{R}}$, there is a unique element $-\alpha \in \tilde{\mathbb{R}}$ such that

$$-[\![\alpha]\!] = [\![-\alpha]\!] \text{ if } \alpha \in \mathbb{R}$$

and $-(-\infty) = \infty$, $-(\infty) = -\infty$. Indeed, $-\alpha$ is the unique element of $\tilde{\mathbb{R}}$ such that whenever $\{a_n\} \in \alpha$, the sequence $\{-a_n\}$ belongs to the class $-\alpha$. This proves (e).

In view of (1), (2) and (3), the bijection $x \to [\![x]\!]$ is an isomorphism of ordered fields. Thus the subset $[\![\mathbb{R}]\!]$ of $\tilde{\mathbb{R}}$ is an isomorphic image of $\mathbb{R}$.

Having completed the construction of a system satisfying (a)—(h) and containing an isomorphic image of $\mathbb{R}$, we now turn our attention to the associative and distributive laws.

Suppose that either none among $\alpha,\beta,\gamma$ is $\infty$ or that none is $-\infty$. Let $\{a_n\} \in \alpha$, $\{b_n\} \in \beta$ and $\{c_n\} \in \gamma$. Then $(\alpha + \beta) + \gamma$ is the unique class in $\tilde{\mathbb{R}}$ containing the sequence $\{(a_n + b_n) + c_n\}$, while $\alpha + (\beta + \gamma)$ is the unique class in $\tilde{\mathbb{R}}$ containing the sequence $\{a_n + (b_n + c_n)\}$. By the associativity of addition in $\mathbb{R}$, it follows that the classes are the same. Thus the equality

$$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$$

holds provided that

either none among $\alpha$, $\beta$, $\gamma$ is $\infty$

or none among $\alpha$, $\beta$, $\gamma$ is $-\infty$.

Similarly, the equality

$$\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$$

holds provided that

either none among $\beta$, $\gamma$, $\alpha\beta$, $\alpha\gamma$ is $\infty$

or none among $\beta$, $\gamma$, $\alpha\beta$, $\alpha\gamma$ is $-\infty$.

In fact, both sides of the equality are the unique class containing the sequence $\{a_n(b_n + c_n)\}$, where $\{a_n\} \in \alpha$, $\{b_n\} \in \beta$ and $\{c_n\} \in \gamma$. It is left to the reader to formulate the corresponding statement regarding $(\alpha\beta)\gamma = \alpha(\beta\gamma)$, which is valid without any restrictions on $\alpha,\beta,\gamma$.

We conclude with two remarks.

1. If $[\![x]\!]$ were enlarged to include all real sequences converging to $x$, then the properties (a)—(g) would follow in the same manner as above, but $\pm\infty \cdot 0$ and $0 \cdot (\pm\infty)$ would remain undefined.

2. If in the construction of $\mathbb{R}$ by Dedekind cuts as in [1, pp.17-21] or [3, pp.47-52], one includes the empty set and $\mathbb{Q}$ as cuts, then one gets $\tilde{\mathbb{R}}$, with the empty set serving as $-\infty$ and $\mathbb{Q}$ as $\infty$. The additional effort involved in checking this is minimal. However, the sum $-\infty + \infty$ needs to be specifically excluded in the general definition of sum, because otherwise it works out to be $-\infty$.

## References

[1] Rudin, W., Principles of Mathematical Analysis, 3rd ed., McGraw-Hill, New York, 1976

[2] Rudin, W., Real and Complex Analysis, McGraw-Hill, New York, 1966

[3] Shirali, S. and Vasudeva, H.L., Mathematical Analysis, Alpha Science Publishers, Oxford, 2006.